NSG-254-62

UNPUBLISHED PRELIMINARY DATA

FACILITY FORM 602

N64-27871	
(ACCESSION NUMBER)	(THRU)
284	1
(PAGES)	(CODE)
CR-56895	29
(NASA CR OR TMX OR AD NUMBER)	(CATEGORY)

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

284 p

VOLUME II — APPENDICES

INTERPLANETARY MIDCOURSE GUIDANCE ANALYSIS

by

Robert Gottlieb Stern

B.S., Lehigh University, 1941

M.S., Stevens Institute of Technology, 1950

Doctor of Science

TE-5

EXPERIMENTAL ASTRONOMY LABORATORY

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

CAMBRIDGE 39, MASSACHUSETTS

OTS PRICE

XEROX $ 18.00 ph

MICROFILM $ ______

VOLUME II - APPENDICES

INTERPLANETARY MIDCOURSE GUIDANCE ANALYSIS

by

ROBERT GOTTLIEB STERN

B.S., Lehigh University, 1941
M.S., Stevens Institute of Technology, 1950

SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE
DEGREE OF DOCTOR OF SCIENCE

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

TABLE OF CONTENTS

VOLUME I

TABLE OF CONTENTS (Cont.)

VOLUME I

TABLE OF CONTENTS (Cont.)

VOLUME I

TABLE OF CONTENTS (Cont.)

VOLUME I

LIST OF ILLUSTRATIONS
VOLUME I

LIST OF ILLUSTRATIONS (Cont.)

VOLUME I

LIST OF TABLES
VOLUME I

TABLE OF CONTENTS

VOLUME II

TABLE OF CONTENTS (Cont.)

VOLUME II

TABLE OF CONTENTS (Cont.

VOLUME II

TABLE OF CONTENTS (Cont.)
VOLUME II

Appendix

TABLE OF CONTENTS (Cont.)
VOLUME II

Appendix

TABLE OF CONTENTS (Cont.)
VOLUME II

LIST OF ILLUSTRATIONS
VOLUME II

Figure

LIST OF ILLUSTRATIONS (Cont.)

VOLUME II

Figure

LIST OF TABLES
VOLUME II

APPENDIX A
COORDINATE SYSTEMS

A.1 Summary

Judicious choice of a coordinate system is of primary importance in the analysis of two-body motion. Three basic systems, each related to the space vehicle's nominal, or reference, trajectory, are defined in this appendix. The designations of the three basic systems are:

1. Reference trajectory stationary coordinate system
2. Reference trajectory local vertical coordinate system
3. Reference trajectory flight path coordinate system

The orientation of these three systems is specified with respect to the conventional heliocentric ecliptic coordinate system, which is also defined.

A.2 Heliocentric Ecliptic Coordinate System

The heliocentric ecliptic axis system is one of the standard systems in celestial mechanics. Its origin is at the center of the sun. Its axes are designated x_E, y_E, and z_E. The x_E and y_E axes are in the ecliptic plane. The x_E -axis lies along the intersection of the equatorial plane with the ecliptic plane, with the positive direction being the direction of the sun from the earth at the time of the vernal equinox (or the direction of the earth from the sun at the time of the autumnal equinox). The positive y_E -axis is obtained by rotating the positive x_E -axis 90^o in the direction of the earth's rotation about the sun. The z_E -axis is normal to the ecliptic plane and positive in the direction of the angular momentum vector of the earth's motion with respect to the sun.

A.3 Reference Trajectory Stationary Coordinate System

The reference trajectory stationary coordinate system, with axes x, y, and z, is related to the nominal two-body path of the space vehicle in the sun's gravitational field. The origin is at the center of the sun. The x-y plane is the plane containing the vehicle's reference trajectory. The positive x-axis is in the direction of perihelion from the sun. The y-axis lies along the latus rectum; its positive direction is obtained by rotating the positive x-axis 90° in the direction of the motion of the vehicle around the sun. The positive z-axis is in the direction of the angular momentum vector of the vehicle's motion relative to the sun.

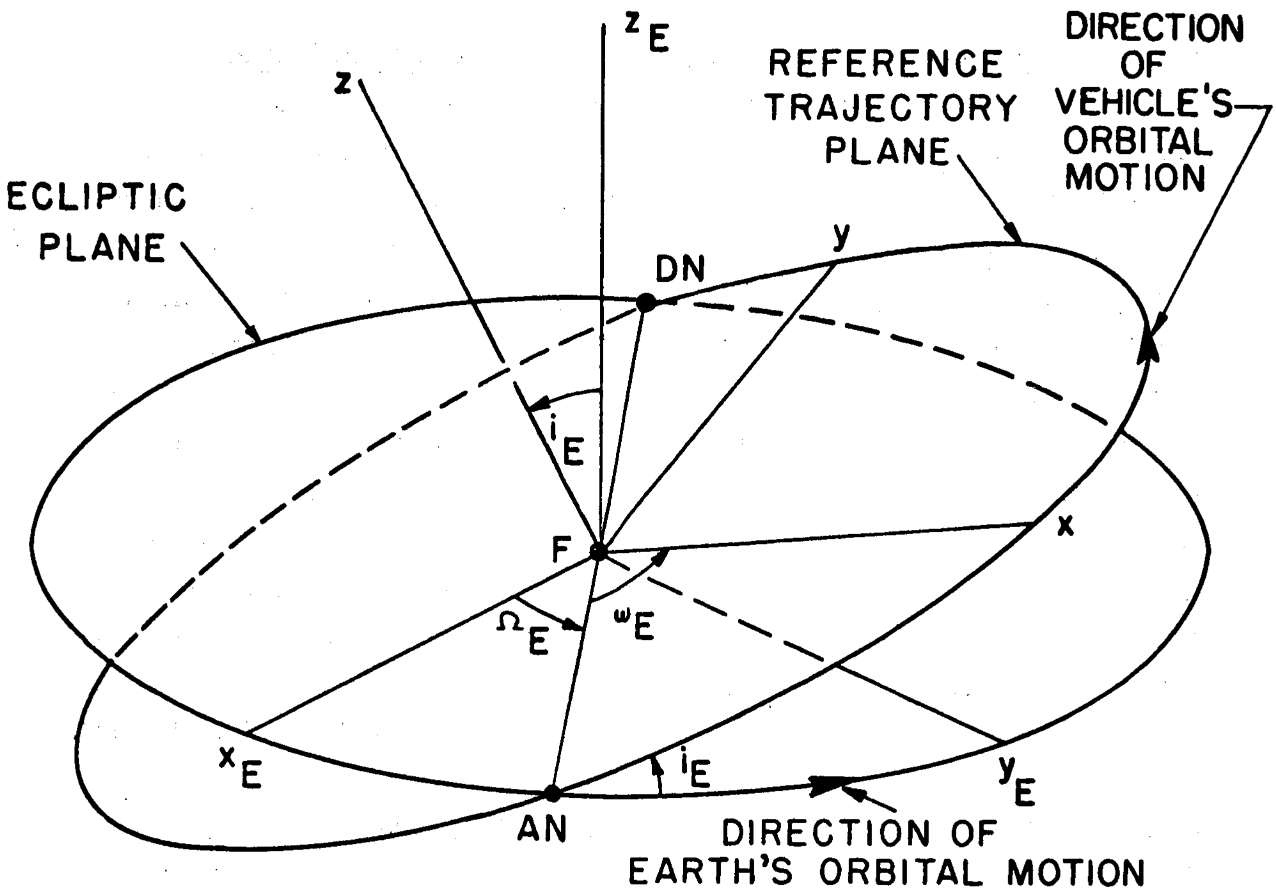
The x, y, z axes may be located with respect to the x_E, y_E, z_E axes by means of the three Euler angles Ω_E, i_E, and ω_E. Ω_E is the longitude of the ascending node. It is the angle, measured in the x_E-y_E plane, between the x_E-axis and the positive half of the line of nodes. The line of nodes is the line of intersection between the ecliptic plane and the reference trajectory plane. The ascending node, which lies on the positive half of the line of nodes, is the point at which the vehicle passes through the ecliptic plane in the direction of increasing z_E.

i_E is the inclination angle. It is the angle subtended at the line of nodes between the reference trajectory plane and the ecliptic plane. It is also the angle between the z-axis and the z_E-axis. The range of i_E is 0° to 180°.

ω_E is the latitude of perihelion. It is the angle, measured in the reference trajectory plane, between the positive half of the line of nodes and the positive x-axis.

The sum of Ω_E and ω_E is known as the longitude of perihelion and is designated ϕ_E. ϕ_E is sometimes referred to as a "broken" angle because its two constituent parts lie in different planes. ϕ_E may be substituted for either Ω_E or ω_E in locating the x, y, z axes.

The angles Ω_E, i_E, and ω_E are illustrated in Fig. A.1.



F – origin at center of sun
AN – ascending node
DN – descending node
x_E, y_E, z_E – ecliptic system coordinate axes
x,y,z – reference trajectory stationary system coordinate axes
Ω_E – longitude of ascending node
i_E – inclination of reference trajectory plane
ω_E – latitude of perihelion of reference trajectory

Figure A.1 Euler Angles Ω_E, i_E, ω_E

A. 4 Reference Trajectory Local Vertical Coordinate System

The reference trajectory local vertical coordinate system, with axes r, s, and z, has its origin at the center of the sun, and its positive z direction lies along the angular momentum vector of the vehicle's motion with respect to the sun. In these two respects it is the same as the reference trajectory stationary system. Also, the r-s plane coincides with the x-y plane. The two systems differ in that the r and s axes rotate in the reference trajectory plane, with the positive direction of the r-axis at any given time lying in the direction of the nominal position of the vehicle at that time. The positive s-axis is 90^{o} "ahead" (i.e., rotated in the direction of vehicle motion) of the positive r-axis.

The angle between the r-axis and the x-axis at any instant is the true anomaly f. Thus, the local vertical system is rotating about the z-axis with angular velocity $\dot{f}$.

The positive r direction will be referred to as the radial direction; similarly, the positive s direction is the transverse direction, and the positive z direction is the orthogonal direction. The r direction is the direction of the vehicle's local vertical in the sun's gravitational field.

Because of the way in which the axes are defined, the values of s and z on a two-body reference trajectory are identically zero for all values of time.

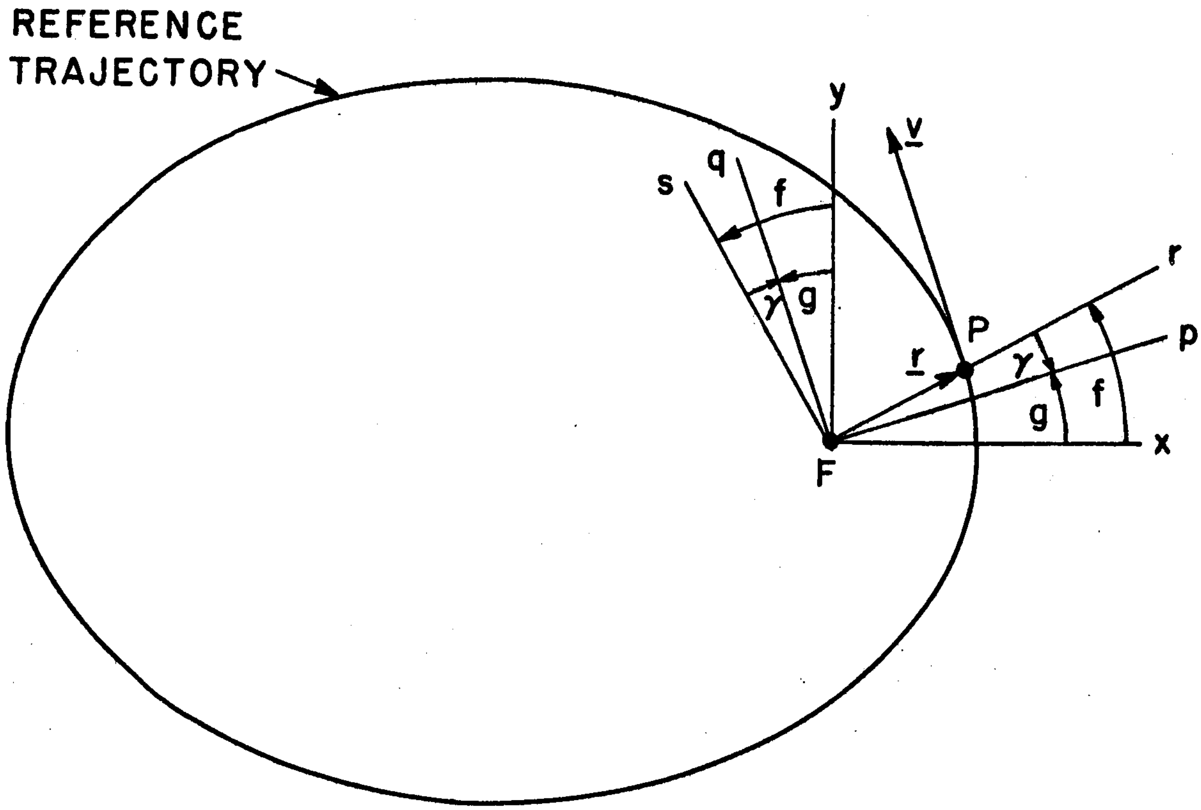
A. 5 Reference Trajectory Flight Path Coordinate System

The axes of the reference trajectory flight path coordinate system are designated p, q, and z. Like the previous two reference trajectory systems, this system has its origin at the center of the sun and its positive z-axis in the direction of the angular momentum vector of the vehicle's motion about the sun. The p-q plane is the reference trajectory plane. The positive q-axis is parallel to the relative velocity vector of the vehicle's nominal motion with respect to the sun. The positive p-axis is 90^{o} "behind" (i.e., rotated in the direction opposite to the vehicle's motion about the sun). the positive q-axis.

The angle between the s-axis and the q-axis is γ, the flight path angle. The angle is positive when the positive q-axis lies between the positive directions of the r and s axes. Since the s-axis represents the "horizontal" direction in the reference trajectory plane, γ is the inclination of the flight path to the horizontal.

The angle between the p-axis and the x-axis is g; it is equal to the difference between f and γ. The angular velocity of the p, q, z coordinate system about the z-axis is $\dot{g}$; which is equal to $(\dot{f} - \dot{\gamma})$.

The orientations of the axes of the three reference trajectory coordinate systems in the reference trajectory plane are shown in Fig. A. 2.



Fq is parallel to $\underline{v}$.

F – attractive focus (center of sun)

P – vehicle position on reference trajectory

$\underline{r}$ – position vector

$\underline{v}$ – velocity vector

x,y – stationary system coordinate axes

r,s – local vertical system coordinate axes

p,q – flight path system coordinate axes

$f = \sphericalangle xFr = \sphericalangle yFs$ = true anomaly

$\gamma = \sphericalangle rFp = \sphericalangle sFq$ = flight path angle

$g = \sphericalangle xFp = \sphericalangle yFq = f - \gamma$

Figure A.2 Orientations of Reference Trajectory Coordinate Systems

APPENDIX B
CELESTIAL MECHANICS

B.1 Summary

Some of the more important relations in celestial mechanics are stated, with particular emphasis on those applicable to elliptical orbits. These relations form the foundation on which much of the subsequent analysis is based. Since all of this material is well known, no attempt is made to supply formal proofs of the equations presented. Such proofs may be found in any standard textbook on this subject, for example, in Chapters 1 and 2 of Smart[28].

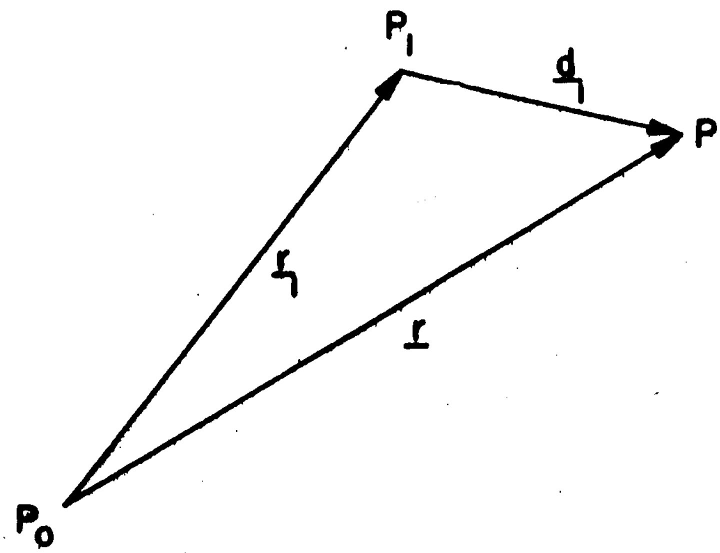
B.2 Motion of a Small Mass in a Many-Body Gravitational Field

Figure B.1 shows the relative positions of three bodies, P_0, P, and P_1. The motion of P is to be investigated under the assumption that the only forces acting on P are those due to the gravitational effects of P_0 and P_1.

For a space vehicle on an interplanetary voyage, P_0 represents the sun, P represents the vehicle, and P_1 normally represents one of the planets.

The vector form of the equation of motion of P is

$$\ddot{\underline{r}} + \frac{\mu}{r^3}\underline{r} = - G\, m_1 \left(\frac{1}{d_1^{\,3}}\underline{d}_1 + \frac{1}{r_1^{\,3}}\underline{r}_1\right) \tag{B-1}$$



P, P_o, P_1 – three bodies treated as hypothetical point-masses

P – body whose motion is being investigated

P_o, P_1 – bodies whose masses affect the motion of P

$\underline{r}, \underline{r}_1, \underline{d}_1$ – position vectors

Figure B.1 Vector Diagram for the Three-Body Problem

The vectors $\underline{r}$, $\underline{r}_1$, and $\underline{d}_1$ are the position vectors of Fig. B.1, with r, r_1, and d_1 being their respective magnitudes. $\ddot{\underline{r}}$ is the inertial acceleration vector of P with respect to P_0.

The masses of P_0, P, and P_1 are m_0, m, and m_1, respectively. The quantity μ is defined by

$$\mu = G (m_0 + m) \tag{B-2}$$

where G is the constant of gravitation.

Since m_0 is the mass of the sun and m_1 is the mass of a planet, which is very much smaller, the motion of P is due primarily to P_0, with P_1 exerting a relatively minor effect (unless the magnitude of $\underline{r}$ is much greater than that of $\underline{d}_1$). In astronomical parlance, the force exerted by P_1 on P is known as the "disturbing force", and the effect of P_1 on the motion of P is known as a "perturbation".

In general, there may be many disturbing forces, due to planets P_1, P_2, ·······, P_n. The vector equation of motion when there are n disturbing forces is

$$\ddot{\underline{r}} + \frac{\mu}{r^3} \underline{r} = - G \sum_{i=1}^{n} m_i \left(\frac{1}{d_i^3} \underline{d}_i + \frac{1}{r_i^3} \underline{r}_i\right) \tag{B-3}$$

B.3 Equations of Motion in Reference Trajectory Coordinate Systems

The vector Eq. (B-3) is a compact form for three component equations, which can be written in any convenient coordinate system. In this section the component equations will be written in the three reference trajectory coordinate systems described in Appendix A.

In the x y z system,

$$\underline{r} = x\,\underline{u}_x + y\,\underline{u}_y + z\,\underline{u}_z \tag{B-4}$$

$$\underline{v} = \dot{\underline{r}} = \dot{x}\,\underline{u}_x + \dot{y}\,\underline{u}_y + \dot{z}\,\underline{u}_z \tag{B-5}$$

$$\underline{a} = \ddot{\underline{r}} = \ddot{x}\,\underline{u}_x + \ddot{y}\,\underline{u}_y + \ddot{z}\,\underline{u}_z \tag{B-6}$$

The symbol $\underline{u}$ represents a unit vector, with the appended subscript indicating its direction. $\underline{v}$ and $\underline{a}$ are, respectively, the inertial velocity and the inertial acceleration of the body P.

$$r^2 = x^2 + y^2 + z^2 \tag{B-7}$$

$$d_i^2 = (x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2 \tag{B-8}$$

$$r_i^2 = x_i^2 + y_i^2 + z_i^2 \tag{B-9}$$

The three component equations of motion may be written in matrix form as follows:

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} + \frac{\mu}{r^3} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -G \sum_{i=1}^{n} m_i \left[\frac{1}{d_i^3} \begin{pmatrix} x - x_i \\ y - y_i \\ z - z_i \end{pmatrix} + \frac{1}{r_i^3} \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix} \right] \tag{B-10}$$

In the r s z coordinate system, the projection of the vector $\underline{r}$ in the r-s plane is designated ρ. The r-axis lies along the projection of $\underline{r}$ in the r-s plane. The coordinate system rotates about the z-axis with angular velocity $\dot{f}$.

$$\underline{r} = \rho\, \underline{u}_r + z\, \underline{u}_z \tag{B-11}$$

$$\underline{v} = \dot{\rho}\, \underline{u}_r + \rho\, \dot{f}\, \underline{u}_s + \dot{z}\, \underline{u}_z \tag{B-12}$$

$$\underline{a} = (\ddot{\rho} - \rho\, \dot{f}^2)\, \underline{u}_r + (\rho\, \ddot{f} + 2\dot{\rho}\, \dot{f})\, \underline{u}_s + \ddot{z}\, \underline{u}_z \tag{B-13}$$

$$r^2 = \rho^2 + z^2 \tag{B-14}$$

$$d_i^2 = (\rho - \rho_i)^2 + s_i^2 + (z - z_i)^2 \tag{B-15}$$

$$r_i^2 = \rho_i^2 + s_i^2 + z_i^2 \tag{B-16}$$

The component equations are

$$\begin{pmatrix} \ddot{\rho} - \rho\dot{f}^2 \\ \rho\, \ddot{f} + 2\dot{\rho}\, \dot{f} \\ \ddot{z} \end{pmatrix} + \frac{\mu}{r^3}\begin{pmatrix} \rho \\ 0 \\ z \end{pmatrix} = - G \sum_{i=1}^{n} m_i \left[\frac{1}{d_i^3}\begin{pmatrix} \rho - \rho_i \\ - s_i \\ z - z_i \end{pmatrix} + \frac{1}{r_i^3}\begin{pmatrix} \rho_i \\ s_i \\ z_i \end{pmatrix} \right] \tag{B-17}$$

The p q z coordinate system rotates about the **z**-axis with angular velocity $\dot{g}$. The q-axis is parallel to the projection of $\underline{v}$ in the p-q plane.

$$\underline{r} = p\,\underline{u}_p + q\,\underline{u}_q + z\,\underline{u}_z \tag{B-18}$$

$$\underline{v} = (\dot{p} - q\,\dot{g})\,\underline{u}_p + (\dot{q} + p\,\dot{g})\,\underline{u}_q + \dot{z}\,\underline{u}_z \tag{B-19}$$

$$= v_q\,\underline{u}_q + \dot{z}\,\underline{u}_z \tag{B-20}$$

The angular velocity $\dot{g}$ may be expressed in terms of $\dot{p}$ and q by equating coefficients of $\underline{u}_p$ in (B-19) and (B-20).

$$\dot{p} - q\,\dot{g} = 0 \tag{B-21}$$

$$\dot{g} = \frac{\dot{p}}{q} \tag{B-22}$$

The acceleration $\underline{a}$ is given by

$$\underline{a} = -\,\dot{g}\,v_q\,\underline{u}_p + \dot{v}_q\,\underline{u}_q + \ddot{z}\,\underline{u}_z \tag{B-23}$$

$$= -\,\frac{\dot{p}\,(p\,\dot{p} + q\,\dot{q}\,)}{q^2}\,\underline{u}_p$$

$$+ \left[\frac{p\,\ddot{p} + q\,\ddot{q}}{q} + \frac{\dot{p}\,(\dot{p}\,q - \dot{q}\,p)}{q^2}\right]\underline{u}_q + \ddot{z}\,\underline{u}_z \tag{B-24}$$

The distance equations are

$$r^2 = p^2 + q^2 + z^2 \tag{B-25}$$

$$d_i^2 = (p - p_i)^2 + (q - q_i)^2 + (z - z_i)^2 \tag{B-26}$$

$$r_i^2 = p_i^2 + q_i^2 + z_i^2 \tag{B-27}$$

The equations of motion in the p q z system are

$$\begin{pmatrix} \dfrac{-\dot{p}\,(p\,\dot{p} + q\,\dot{q})}{q^2} \\[2ex] \dfrac{p\,\ddot{p} + q\,\ddot{q}}{q} + \dfrac{\dot{p}\,(\dot{p}\,q - \dot{q}\,p)}{q^2} \\[2ex] \ddot{z} \end{pmatrix} + \frac{\mu}{r^3} \begin{pmatrix} p \\ q \\ z \end{pmatrix}$$

$$= - G \sum_{i=1}^{n} m_i \left[\frac{1}{d_i^3} \begin{pmatrix} p - p_i \\ q - q_i \\ z - z_i \end{pmatrix} + \frac{1}{r_i^3} \begin{pmatrix} p_i \\ q_i \\ z_i \end{pmatrix} \right] \tag{B-28}$$

B.4 Two-Body Motion

When there are no disturbing forces, the motion of P is the classic two-body motion, and the vector equation reduces to

$$\ddot{\underline{r}} + \frac{\mu}{r^3}\,\underline{r} = \underline{0}_3 \tag{B-29}$$

The acceleration vector and the position vector are now collinear. Therefore, the motion of P must lie wholly within the plane determined by the position vector and the velocity vector existing at any specified time.

The component equations of motion in the x y z coordinate system are

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} + \frac{\mu}{r^3} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{B-30}$$

If the axes are so chosen that z is perpendicular to the plane of the two-body motion, z is always zero, and the motion of P is completely described by the first two equations of (B-30). The distance r is then given by

$$r^2 = x^2 + y^2 \tag{B-31}$$

In the r s z coordinate system, ρ becomes equal to r for two-body motion. The equations of motion in the trajectory plane are

$$\begin{pmatrix} \ddot{r} - r\dot{f}^2 \\ r\ddot{f} + 2\dot{r}\dot{f} \end{pmatrix} + \frac{\mu}{r^3} \begin{pmatrix} r \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{B-32}$$

When the p q z coordinate system is used to describe the two-body motion,

$$r^2 = p^2 + q^2 \tag{B-33}$$

The equations of motion in the trajectory plane are

$$\begin{pmatrix} \dfrac{-\dot{p}\,(p\,\dot{p} + q\,\dot{q})}{q^2} \\ \dfrac{p\,\ddot{p} + q\,\ddot{q}}{q} + \dfrac{\dot{p}\,(\dot{p}\,q - \dot{q}\,p)}{q^2} \end{pmatrix} + \frac{\mu}{r^3} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{B-34}$$

B. 5 Integration of Equations of Two-Body Motion

The integration of the equations of two-body motion is most easily accomplished by using (B-32). The lower equation of (B-32) may be integrated directly, with the result

$$r^2\,\dot{f} = h \tag{B-35}$$

where h, a constant, is the angular momentum of P per unit mass.

To integrate the upper equation of (B-32), f is substituted for t as the independent variable, and the dependent variable r is replaced by u, where

$$u = \frac{1}{r} \tag{B-36}$$

In terms of u and f, the upper equation of (B-32) becomes

$$\frac{d^2 u}{d f^2} + u = \frac{\mu}{h^2} \tag{B-37}$$

The solution for r is

$$r = \frac{\frac{h^2}{\mu}}{1 + e \cos (f - \omega)} \tag{B-38}$$

where e and ω are constants of integration.

Equation (B-38) is the polar-coordinate form of the equation of a general conic section, with the origin at one focus. The constant e is the eccentricity of the conic. ω is the angle between the arbitrarily chosen x-axis in the x-y plane and the major axis of the conic. If new x and y axes are defined such that the new x-axis coincides with the major axis of the conic, then the angle between the new axes and the old axes is ω, and $(f - \omega)$ may be replaced by f in (B-38). The new angle f, measured from the new x-axis, is the true anomaly.

B. 6 Orbital Elements

The component equations of (B-30), the general equations of motion of the two-body problem, are three second-order linear differential equations, and consequently their complete solution involves six arbitrary constants. The six constants may be the three components of position and the three components of velocity occurring at a specified time, or they may be three components of position at each of two specified times. There are many other groupings of six constants that may be used.

A grouping that is widely used in celestial mechanics is one known as the six orbital elements. These elements are:

1. a, The semi-major axis of the conic section
2. e, The eccentricity of the conic section
3. Ω, The longitude of the ascending node
4. i, The inclination of the trajectory plane
5. ω, The latitude of perihelion
6. t_0, The time of perihelion passage

The elements a and e determine the size and shape, respectively, of the conic section.

The angles Ω and i determine the orientation of the trajectory plane, and angle ω locates the axes of the conic section in the trajectory plane. If the standard coordinate system to which the three angles are referred is the heliocentric ecliptic system, the three become Ω_E, i_E, and ω_E, which are defined in Section A. 3 and illustrated in Fig. A. 1.

The element t_0 relates position on the trajectory to some arbitrarily chosen time reference, known as the epoch; t_0 is the time, relative to the epoch, at which the vehicle passes through the perihelion point.

Choices other than these given above may be made for the orbital elements. Obviously, any choice of a new element may be expressed as a combination of those elements already listed.

By convention, the range of e is limited to zero to infinity, while a may take on any value from minus infinity to plus infinity. The basic form of a particular conic section is determined by the values of e and a associated with it. There are three basic forms, hyperbolas, parabolas, and ellipses. If e is greater than one and a is negative, the trajectory is hyperbolic; if e equals one and a is infinite, the trajectory is parabolic; if e is less than one and a is positive, the trajectory is elliptical.

In the present analysis, which is intended to be applicable primarily to the midcourse phase of interplanetary voyages, only elliptical forms are considered in detail.

B.7 Geometric Properties of the Ellipse

The polar form of the equation of a conic section, with the origin at one focus, is

$$r = \frac{\ell}{1 + e \cos f} \tag{B-39}$$

where the constant ℓ is the semi-latus rectum. ℓ is the value of r corresponding to

$$f = \pm \frac{\Pi}{2}$$

In terms of a and e,

$$\ell = a\,(1 - e^2) \tag{B-40}$$

When the conic section is an ellipse, its equation in rectangular coordinates, with origin at one focus, is

$$\frac{(x + c)^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{B-41}$$

b is the semi-minor axis of the ellipse.

$$b = a\,(1 - e^2)^{1/2} \tag{B-42}$$

The linear eccentricity c is defined by

$$c = a\,e \tag{B-43}$$

c is the distance along the major axis from the center of the ellipse to either focus. The lengths a, b, and c are related by the equation

$$a^2 = b^2 + c^2 \tag{B-44}$$

The sum of the distances of any point on the ellipse from each of the two foci is equal to 2 a.

The quantities introduced in this section are shown in Fig. B. 2.

B. 8 The Anomalies

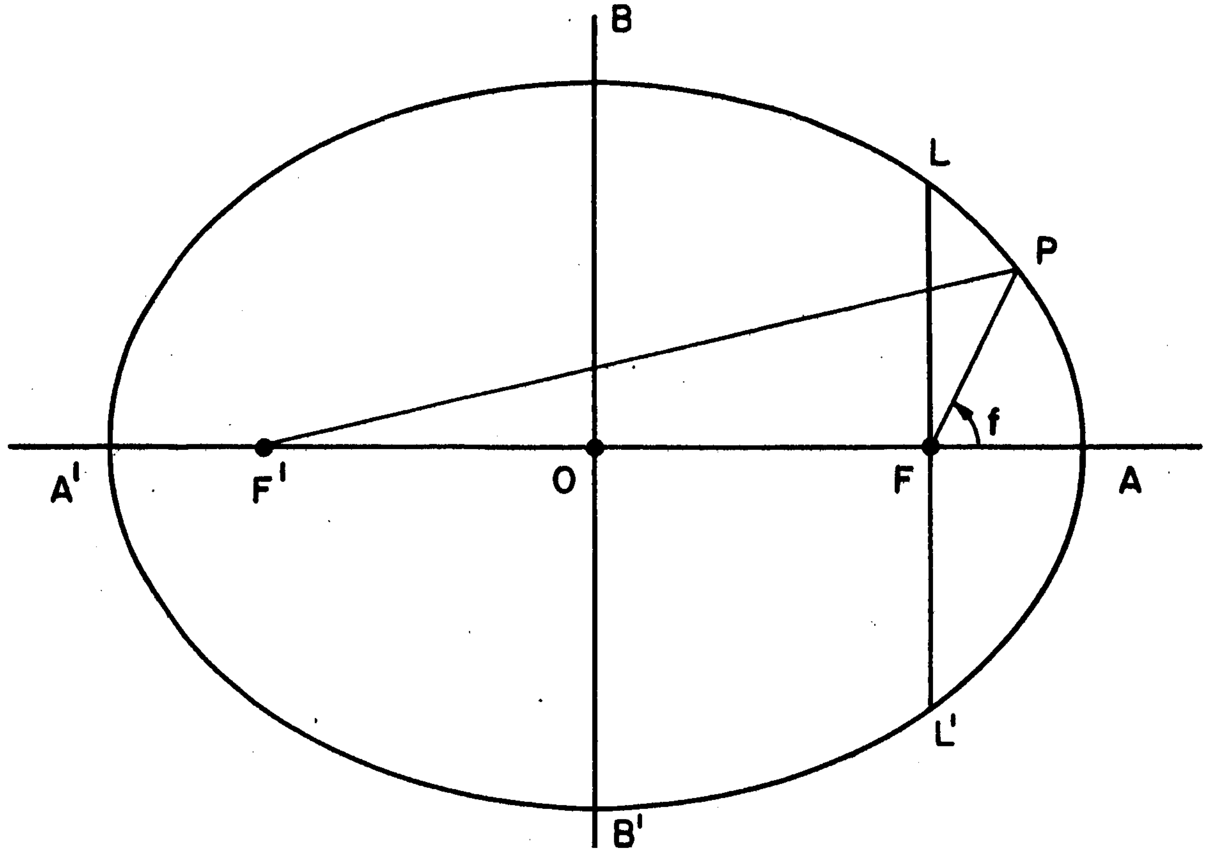
The true anomaly has been introduced in Section B. 5. Two other anomalies that are widely used in celestial mechanics are the eccentric anomaly E and the mean anomaly M.

The geometric construction required to obtain the eccentric anomaly is indicated in Fig. B. 3. The eccentric anomaly is related to the circle of radius a circumscribed about the ellipse whose semi-major axis is a.

The mean anomaly varies linearly with elapsed time t.

$$M = n\,(t - t_0) \tag{B-45}$$

where n is a constant known as the mean angular motion. n is the average angular velocity of the space vehicle in its elliptical orbit



O – center of ellipse

F, F' – foci of ellipse

A'OA – major axis

BOB' – minor axis

LFL' – latus rectum

P – arbitrary point on ellipse

OA = OA' = a = semi-major axis

OB = OB' = b = semi-minor axis

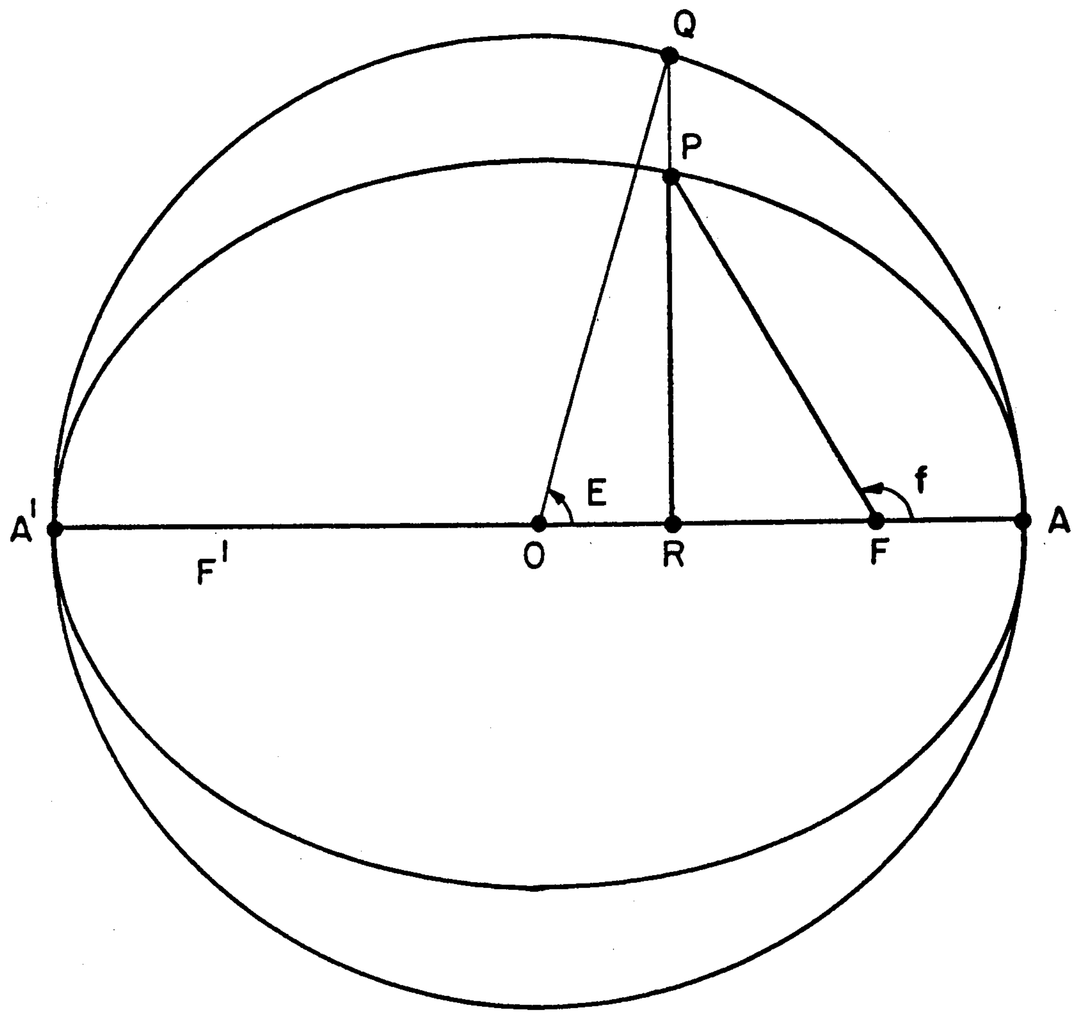
FL = FL' = ℓ = semi-latus rectum

OF = OF' = c = linear eccentricity

$\sphericalangle$ AFP = f = true anomaly

F'P + PF = 2a

Figure B.2 The Ellipse



APA' – elliptical arc with semi-major axis a

AQA' – circular arc of radius a

O – center of ellipse and of circle

F,F' – foci of ellipse

P – arbitrary point on ellipse

QPR $\perp$ A'A

$\sphericalangle$AFP = f = true anomaly

$\sphericalangle$AOQ = E = eccentric anomaly

Figure B.3 Graphical Construction of Eccentric Anomaly about the sun.

$$n = \frac{2\Pi}{P} \qquad \text{(B-46)}$$

where P is the period of the trajectory.

The constant t_0 in Eq. (B-45) is the time of perihelion passage, the sixth orbital element of Section B.5.

An alternate form for Eq. (B-45) is

$$M = n\,t + M_0 \qquad \text{(B-47)}$$

where $M_0 = -\,n\,t_0$ (B-48)

M_0 is the value of the mean anomaly at time $t = 0$. M_0 is sometimes used in place of t_0 as one of the orbital elements.

The true anomaly and the eccentric anomaly are related by the following series of equations:

$$r = \frac{a\,(1 - e^2)}{1 + e\cos f} = a\,(1 - e\cos E) \qquad \text{(B-49)}$$

$$x = r\cos f = a\,(\cos E - e) \qquad \text{(B-50)}$$

$$y = r\sin f = a\,(1 - e^2)^{1/2}\sin E \qquad \text{(B-51)}$$

$$(1 + e\cos f)\,(1 - e\cos E) = 1 - e^2 \qquad \text{(B-52)}$$

$$\sin f = \frac{(1 - e^2)^{1/2}\sin E}{1 - e\cos E} \qquad \sin E = \frac{(1 - e^2)^{1/2}\sin f}{1 + e\cos f} \qquad \text{(B-53)}$$

$$\cos f = \frac{\cos E - e}{1 - e\cos E} \qquad \cos E = \frac{\cos f + e}{1 + e\cos f} \qquad \text{(B-54)}$$

The eccentric anomaly and the mean anomaly are related through Kepler's equation.

$$M = E - e \sin E \tag{B-55}$$

The eccentric anomaly serves as a bridge relating the geometric variable f to the dynamic variable M (or t).

B.9 Dynamic Relations for Elliptical Trajectories

The derivatives of the three anomalies are

$$\dot{M} = n \tag{B-56}$$

$$\dot{E} = \frac{n}{1 - e \cos E} = \frac{n\,(1 + e \cos f)}{1 - e^2} \tag{B-57}$$

$$\dot{f} = \frac{n\,(1 - e^2)^{1/2}}{(1 - e \cos E)^2} = \frac{n\,(1 + e \cos f)^2}{(1 - e^2)^{3/2}} \tag{B-58}$$

It is interesting to note that $\dot{M}$ is equal to a constant, $r\,\dot{E}$ is equal to a constant, and $r^2\,\dot{f}$ is equal to a constant.

$$r\,\dot{E} = n\,a \tag{B-59}$$

$$r^2\,\dot{f} = h = n\,a^2\,(1 - e^2)^{1/2} \tag{B-60}$$

A comparison of Eq. (B-38) with Eq. (B-39) indicates that

$$a\,(1 - e^2) = \frac{h^2}{\mu} \tag{B-61}$$

and therefore,

$$\mu = n^2\,a^3 \tag{B-62}$$

The differentials of E and f may each be expressed in terms of the other.

$$d\,E = \frac{(1 - e^2)^{1/2}\,d\,f}{1 + e\cos f} \tag{B-63}$$

$$d\,f = \frac{(1 - e^2)^{1/2}\,d\,E}{1 - e\cos E} \tag{B-64}$$

The velocity components in the radial and transverse directions may be written in a variety of ways.

$$v_r = \dot{r} = \frac{n\,a\,e\sin E}{1 - e\cos E} = \frac{n\,a\,e\sin f}{(1 - e^2)^{1/2}}$$

$$= \frac{n\,a^2\,e\sin E}{r} = \frac{\mu}{h}\,e\sin f \tag{B-65}$$

$$v_s = r\,\dot{f} = \frac{h}{r} = \frac{n\,a\,(1 - e^2)^{1/2}}{1 - e\cos E} = \frac{n\,a\,(1 + e\cos f)}{(1 - e^2)^{1/2}}$$

$$= \frac{\mu}{h} \cdot \frac{(1 - e^2)}{1 - e\cos E} = \frac{\mu}{h}\,(1 + e\cos f) \tag{B-66}$$

The square of the total orbital velocity is

$$v^2 = \frac{\mu^2}{h^2} (1 + 2 e \cos f + e^2)$$

$$= \mu \left(\frac{2}{r} - \frac{1}{a}\right) \qquad \text{(B-67)}$$

In the literature of celestial mechanics, Eq. (B-67) is known as the "vis viva integral".

The orbital velocity may be expressed in terms of either E or f.

$$v = \frac{\mu}{h} (1 + 2 e \cos f + e^2)^{1/2} \qquad \text{(B-68)}$$

$$= n a \frac{(1 + e \cos E)^{1/2}}{(1 - e \cos E)^{1/2}} \qquad \text{(B-69)}$$

The total energy per unit mass is the sum of the kinetic energy T and the potential energy U.

$$H = T + U = \frac{1}{2} v^2 - \frac{\mu}{r} = - \frac{\mu}{2 a} \qquad \text{(B-70)}$$

The total energy is a function of only one of the six orbital elements, the semi-major axis a.

The velocity components in the x and y directions can also be expressed in many forms.

$$v_x = \dot{x} = - v \sin g$$

$$= - \frac{n\, a \sin E}{1 - e \cos E} = - \frac{n\, a}{(1 - e^2)^{1/2}} \sin f$$

$$= - \frac{n\, a^2 \sin E}{r} = - \frac{\mu}{h} \sin f \qquad \text{(B-71)}$$

$$v_y = \dot{y} = v \cos g$$

$$= \frac{n\, a\, (1 - e^2)^{1/2} \cos E}{1 - e \cos E} = \frac{n\, a}{(1 - e^2)^{1/2}} (\cos f + e)$$

$$= \frac{h}{r} \cos E = \frac{\mu}{h} (\cos f + e) \qquad \text{(B-72)}$$

In the flight path coordinate system the velocity components are simply

$$v_p = 0 \qquad \text{(B-73)}$$

$$v_q = v \qquad \text{(B-74)}$$

The velocity component equations may be used to determine the simple trigonometric functions of γ and g.

$$\sin \gamma = \frac{v_r}{v} = \frac{e \sin E}{(1 - e^2 \cos^2 E)^{1/2}} = \frac{e \sin f}{(1 + 2\, e \cos f + e^2)^{1/2}} \qquad \text{(B-75)}$$

$$\cos \gamma = \frac{v_s}{v} = \frac{(1 - e^2)^{1/2}}{(1 - e^2 \cos^2 E)^{1/2}} = \frac{1 + e \cos f}{(1 + 2 e \cos f + e^2)^{1/2}} \quad \text{(B-76)}$$

$$\sin g = - \frac{v_x}{v} = \frac{\sin E}{(1 - e^2 \cos^2 E)^{1/2}} = \frac{\sin f}{(1 + 2 e \cos f + e^2)^{1/2}} \quad \text{(B-77)}$$

$$\cos g = \frac{v_y}{v} = \frac{(1 - e^2)^{1/2} \cos E}{(1 - e^2 \cos^2 E)^{1/2}} = \frac{\cos f + e}{(1 + 2 e \cos f + e^2)^{1/2}} \quad \text{(B-78)}$$

The angular velocities $\dot{\gamma}$ and $\dot{g}$ are

$$\dot{\gamma} = \frac{n (1 - e^2)^{1/2} e \cos E}{(1 - e \cos E)^2 (1 + e \cos E)} = \frac{n e (1 + e \cos f)^2 (\cos f + e)}{(1 - e^2)^{3/2} (1 + 2 e \cos f + e^2)}$$

(B-79)

$$\dot{g} = \frac{n (1 - e^2)^{1/2}}{(1 - e \cos E)^2 (1 + e \cos E)} = \frac{n (1 + e \cos f)^3}{(1 - e^2)^{3/2} (1 + 2 e \cos f + e^2)}$$

(B-80)

The position components in the flight path system may be expressed in the following ways:

$$p = r \cos \gamma = x \cos g + y \sin g = \frac{h}{v}$$

$$= \frac{a (1 - e^2)^{1/2} (1 - e \cos E)^{1/2}}{(1 + e \cos E)^{1/2}} = \frac{a (1 - e^2)}{(1 + 2 e \cos f + e^2)^{1/2}}$$

(B-81)

$$q = r \sin \gamma = -x \sin g + y \cos g = \frac{h}{v} \tan \gamma$$

$$= \frac{a e \sin E (1 - e \cos E)^{1/2}}{(1 + e \cos E)^{1/2}} = \frac{a (1 - e^2) e \sin f}{(1 + e \cos f)(1 + 2 e \cos f + e^2)^{1/2}} \qquad \text{(B-82)}$$

From (B-66) and (B-81), two alternate forms of the angular momentum equation are

$$h = r v_s = p v \qquad \text{(B-83)}$$

The components of acceleration in the three coordinate systems may be obtained from (B-10), (B-17), and (B-28).

$$a_r = \ddot{r} - r \dot{f}^2 = - \frac{\mu}{r^2} \qquad \text{(B-84)}$$

$$a_s = r \ddot{f} + 2 \dot{r} \dot{f} = 0 \qquad \text{(B-85)}$$

$$a_x = \ddot{x} = - \mu \frac{x}{r^3} \qquad \text{(B-86)}$$

$$a_y = \ddot{y} = - \mu \frac{y}{r^3} \qquad \text{(B-87)}$$

$$a_p = - \dot{g} v = - \frac{\dot{p} (p \dot{p} + q \dot{q})}{q^2} = - \mu \frac{p}{r^3} \qquad \text{(B-88)}$$

$$a_q = \dot{v} = \frac{p \ddot{p} + q \ddot{q}}{q} + \frac{\dot{p} (\dot{p} q - \dot{q} p)}{q^2} = - \mu \frac{q}{r^3} \qquad \text{(B-89)}$$

APPENDIX C
GRAPHICAL CONSTRUCTIONS

C.1 Summary

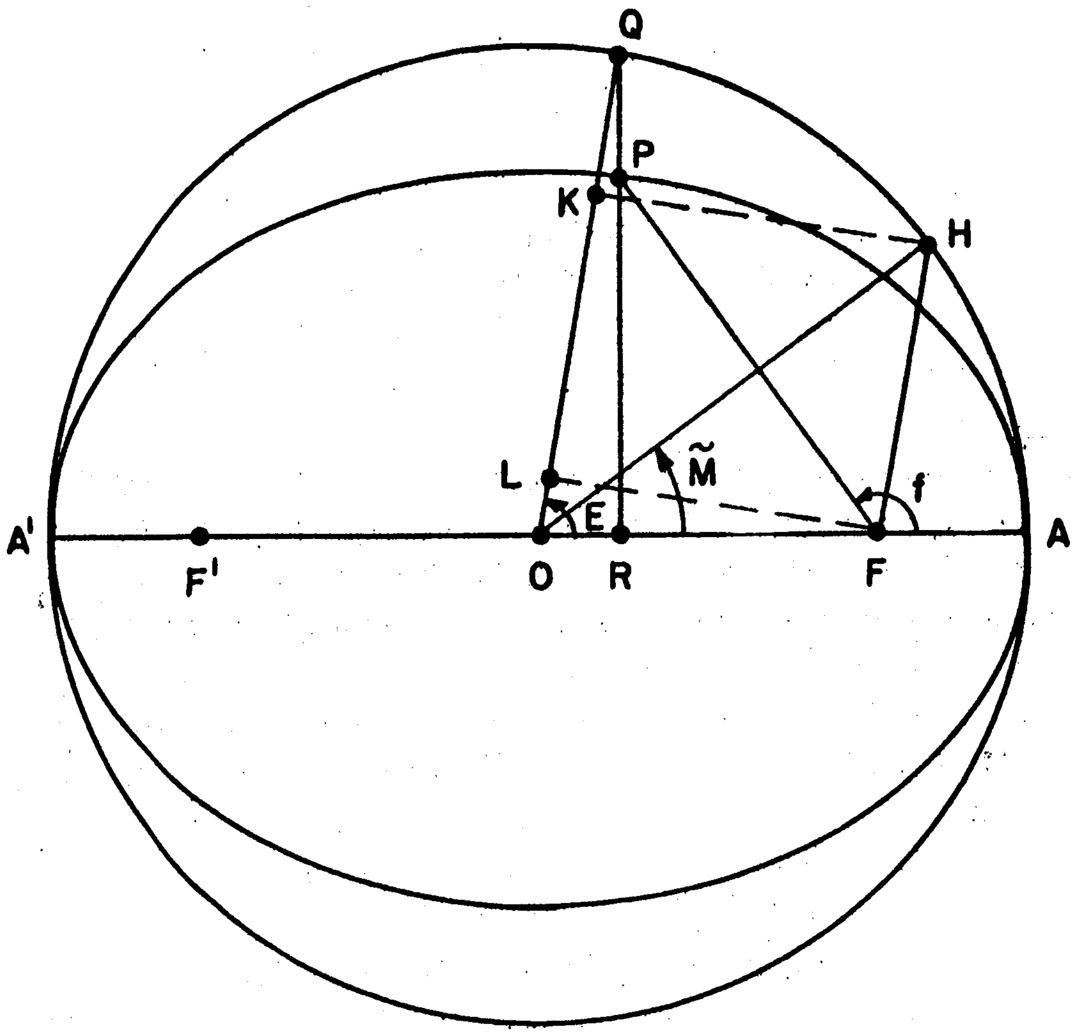
The review of the equations of celestial mechanics in Appendix B has led to the development of two interesting graphical constructions. The first is an approximate representation for the mean anomaly and when the eccentricity e is less the 0.5. The second is an exact method for determining velocity in an elliptical orbit. These constructions are thought to be novel and are presented here as a by-product of the primary analysis with the thought that they may be of value as class-room demonstrations.

C.2 Graphical Representation of Mean Anomaly

The conventional geometric interpretation of the true anomaly and the eccentric anomaly is given in Section B.8 and Fig. B.3. It would be desirable to get a similar representation of the mean anomaly, M, so that one could see graphically the relation between the angular motion and elapsed time. Unfortunately, no simple, exact geometrical construction is known for the mean anomaly. It is the purpose of this section to show a simple, though inexact, method of obtaining the mean anomaly graphically.

Figure C.1, which illustrates the method, is an extension of Fig. B.3. The construction is as follows:

1. From the focus F lay out FH parallel to OQ and meeting the circumscribed circle at H.
2. Connect the center O with point H by a straight line.



$\sphericalangle AFP = f =$ true anomaly

$\sphericalangle AOQ = E =$ eccentric anomaly

$FH \parallel OQ$

$FL \perp OQ$, $HK \perp OQ$, $FL \parallel HK$

$OF = c = ae =$ linear eccentricity

$HK = FL = ae \sin E$

$OA = OH = OQ = a$

$$\sphericalangle HOQ = \sin^{-1}\left(\frac{HK}{OH}\right) = \sin^{-1}(e \sin E)$$

$$\tilde{M} = \sphericalangle AOH = \sphericalangle AOQ - \sphericalangle HOQ = E - \sin^{-1}(e \sin E)$$

= approximation of mean anomaly

Figure C.1 Graphical Approximation of Mean Anomaly

Then angle AOH is equal to $\tilde{M}$, the approximation to the true anomaly.

In order to prove this statement, two auxiliary lines, FL and HK, are drawn. Both of these are perpendicular to OQ, and therefore, they are parallel to each other. From the figure,

$$HK = FL = OF \sin E = ae \sin E \qquad (C\text{-}1)$$

$$\sphericalangle HOQ = \sin^{-1}\left(\frac{HK}{OH}\right) = \sin^{-1}\left(\frac{ae \sin E}{a}\right)$$

$$= \sin^{-1}(e \sin E) \qquad (C\text{-}2)$$

$$\tilde{M} = \sphericalangle AOH = \sphericalangle AOQ - \sphericalangle HOQ$$

$$= E - \sin^{-1}(e \sin E) \qquad (C\text{-}3)$$

The exact equation for the mean anaomaly is

$$M = E - e \sin E \qquad (C\text{-}4)$$

The error in the approximation is

$$\Delta M = \tilde{M} - M = e \sin E - \sin^{-1}(e \sin E) \qquad (C\text{-}5)$$

The maximum magnitude of the error for a given e occurs when

$$E = \pm \frac{\pi}{2}.$$

$$|\Delta M|_{max} = e - \sin^{-1} e \qquad (C\text{-}6)$$

For eccentricities up to 0.4, $|\Delta M|_{max}$ is less than 1°. For $e = 0.5$, it is $1\frac{1}{3}°$.

It may be seen from Fig. C.1 that for

$$0 < f < \pi, \qquad M < E < f \tag{C-7}$$

for

$$\pi < f < 2\pi, \qquad f < E < M \tag{C-8}$$

for

$$f = 0 \quad , \qquad E = 0 = M \tag{C-9}$$

for

$$f = \pi \quad , \qquad E = \pi = M \tag{C-10}$$

C.3 Graphical Solution for Crbital Velocity and Its Components

The objective in this section is to determine graphically the velocity at point P on an elliptical trajectory for which a, e, and n are known.

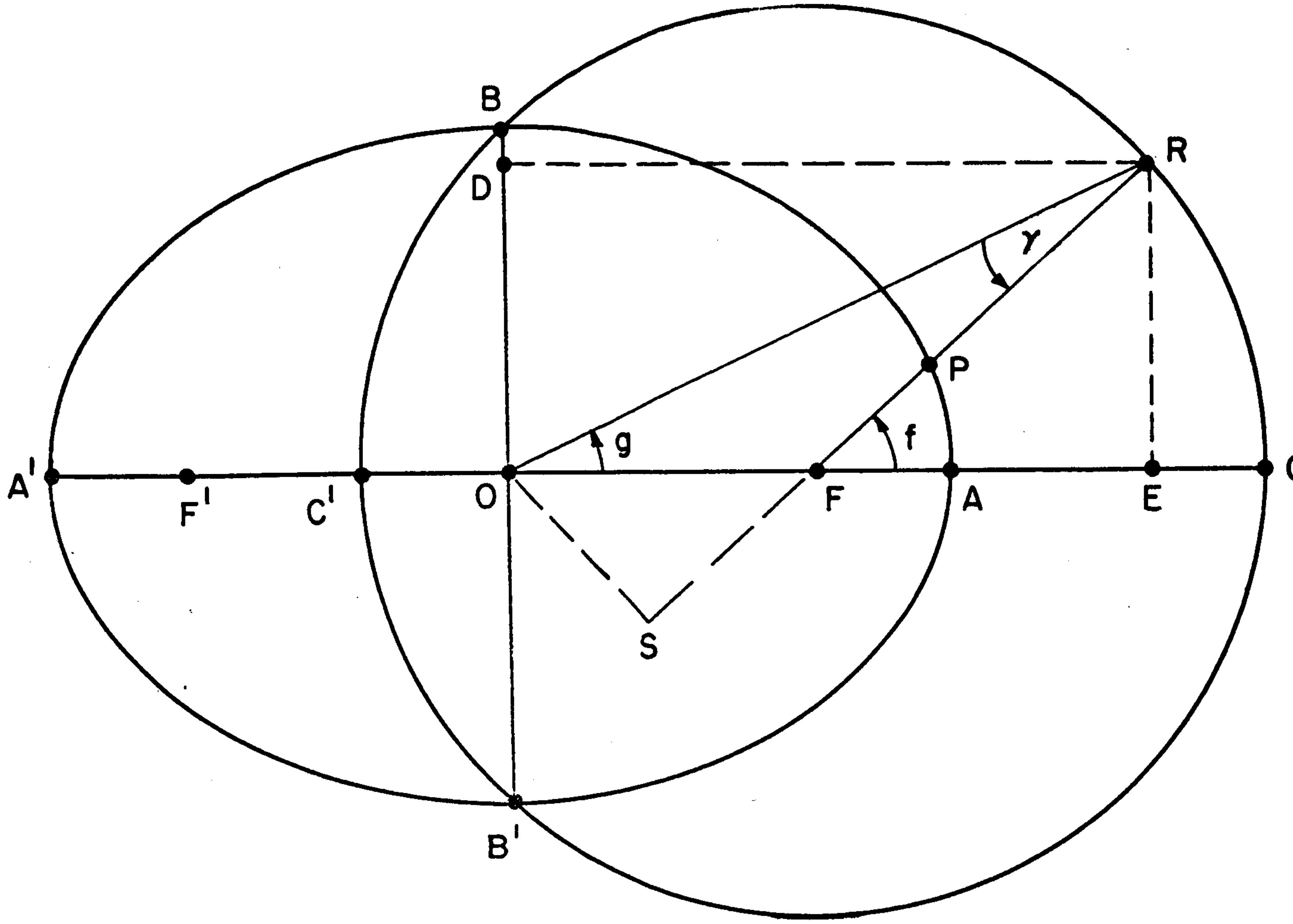
In Fig. C.2, the known trajectory is APBA'B'A. The steps in the graphical construction are the following:

1. With focus F as center and with radius a, describe a circle.
2. Extend the radius vector FP through P until it intersects the circle at R.
3. Draw a straight line connecting the center of the ellipse at O to R.

The length OR is proportional to the orbital velocity at P. The constant of proportionality is

$$\frac{(1 - e^2)^{1/2}}{n}.$$

To prove these statements, several additional lines are drawn. FR is extended through F until it meets the perpendicular dropped from O to the extension of FP. The two lines intersect at S. Then,



APBA'B'A – ellipse with center at O, foci at F and F', and semi-major axis a = OA

CRBC'B'C – circle with center at F and radius a

P – arbitrary point on ellipse

OS ⊥ SR

$$OR = \frac{(1-e^2)^{1/2}}{n} v$$

$$OS = \frac{(1-e^2)^{1/2}}{n} v_r \quad , \quad SR = \frac{(1-e^2)^{1/2}}{n} v_s$$

$$OD = -\frac{(1-e^2)^{1/2}}{n} v_x \quad , \quad OE = \frac{(1-e^2)^{1/2}}{n} v_y$$

$\sphericalangle AFR = f =$ true anomaly

$\sphericalangle ORS = \gamma =$ flight path angle

$\sphericalangle ROA = f - \gamma = g$

Figure C.2 Graphical Determination of Orbital Velocity and Its Components

$$FR = a \tag{C-11}$$

$$OF = a\, e \tag{C-12}$$

$$\sphericalangle AFR = \sphericalangle OFS = f \tag{C-13}$$

In the triangle ORS,

$$SR = a\,(1 + e \cos f) \tag{C-14}$$

$$OS = a\, e \sin f \tag{C-15}$$

$$OR = a\,(1 + 2e \cos f + e^2)^{1/2} \tag{C-16}$$

When the last three equations are compared with Eqs. (B-66), (B-65), and (B-68), it is apparent that

$$SR = \frac{h\,a}{\mu}\, v_s = \frac{(1 - e^2)^{1/2}}{n}\, v_s \tag{C-17}$$

$$OS = \frac{(1 - e^2)^{1/2}}{n}\, v_r \tag{C-18}$$

$$OR = \frac{(1 - e^2)^{1/2}}{n}\, v \tag{C-19}$$

Also,

$$\sphericalangle ORS = \tan^{-1}\left(\frac{OS}{SR}\right) = \tan^{-1}\left(\frac{v_r}{v_s}\right) = \gamma \tag{C-20}$$

$$\sphericalangle AOR = f - \gamma = g \tag{C-21}$$

By constructing the lines RD and RE parallel, respectively, to the major and minor axes of the ellipse, it is easily seen that

$$OD = OR \sin g = \frac{(1 - e^2)^{1/2}}{n} v \sin g = - \frac{(1 - e^2)^{1/2}}{n} v_x \quad (C-22)$$

$$OE = OR \cos g = \frac{(1 - e^2)^{1/2}}{n} v \cos g = \frac{(1 - e^2)^{1/2}}{n} v_y \quad (C-23)$$

The angle between the positive p-axis of the flight path coordinate system and the positive x-axis is g. Therefore, OR in Fig. C.2 has the direction of the p-axis associated with point P on the trajectory. Since the direction of the orbital velocity vector is along the positive q-axis, the direction of $\underline{v}$ may be obtained by rotating OR counter-clockwise through 90 degrees. Similarly, the directions of the velocity components may be found by rotating the corresponding lengths in the figure 90 degrees counter-clockwise.

The circle CRBC'B'C provides a simple means of visualizing the variation of the magnitude of the orbital velocity in an elliptical trajectory. As point P progresses on the ellipse, point R progresses on the circle, and OR is a continuous measure of v.

APPENDIX D
ELLIPTICAL CYLINDRICAL COORDINATES

D.1 Summary

Elliptical cylindrical coordinates are known to be particularly well suited to certain problems involving either ellipses or hyperbolas. Consequently, the applicability of this curvilinear coordinate system to the problem of guiding a vehicle traversing an elliptical trajectory has been investigated.

It is shown that there is an interesting relationship between the elliptical system and the flight path system described in Appendix A. The tangents to the three coordinate curves of the elliptical system are parallel, respectively, to the p, q, and z axes of the flight path system.

A comparison of the elliptical system with the reference trajectory rectilinear systems of Appendix A indicates that the curvilinear system has definite advantages for studying motion along a known, fixed elliptical trajectory. On the other hand, the curvilinear system offers no advantage in the study of the variation of an actual trajectory from a known elliptical reference trajectory. Since the guidance problem is primarily a problem of the latter type, the elliptical cylindrical system has not been used in the ensuing analysis.

D.2 Basic Coordinates in the Elliptical System

The analysis presented below is based on Sections 6.16 and 6.17 of Hildebrand(41); with associated problems 6.25 and 6.26.

Elliptical cylindrical coordinates α, β, z are defined by the equations

$$x_0 = k \cosh \alpha \cos \beta \tag{D-1}$$

$$y = k \sinh \alpha \cos \beta \tag{D-2}$$

$$z = z \tag{D-3}$$

where k is a constant and x_0, y, z are conventional Cartesian coordinates. The reason for the use of x_0 instead of x is explained later in this section.

From (D-1) and (D-2),

$$\frac{x_0^2}{k^2 \cosh^2 \alpha} + \frac{y^2}{k^2 \sinh^2 \alpha} = \cos^2 \beta + \sin^2 \beta = 1 \tag{D-4}$$

If α is a constant, Eq. (D-4) is the equation of an ellipse with the origin at the center of the ellipse. The axes of the ellipse are given by

$$a^2 = k^2 \cosh^2 \alpha \tag{D-5}$$

$$b^2 = a^2 (1 - e^2) = k^2 \sinh^2 \alpha \tag{D-6}$$

k^2 may be determined by subtracting (D-6) from (D-5).

$$k^2 (\cosh^2 \alpha - \sinh^2 \alpha) = k^2 = a^2 e^2 \tag{D-7}$$

With the positive sign being chosen for the square root, k becomes equal to the linear eccentricity

$$k = ae \tag{D-8}$$

By substituting (D-8) into (D-5) and (D-6) and again taking the positive sign for each of the roots, cosh α and sinh α can be expressed in terms of e.

$$\cosh \alpha = \frac{1}{e} \tag{D-9}$$

$$\sinh \alpha = \frac{(1 - e^2)^{\frac{1}{2}}}{e} \tag{D-10}$$

Since the origin of the elliptical coordinate system is at the center of the ellipse rather than at one focus, the quantity x_0 in Eq. (D-1) is not the same as x in the Cartesian system of Appendix A. The equation relating x to x_0 is

$$x = x_0 - ae \tag{D-11}$$

The coordinates y and z in Eqs. (D-2) and (D-3) are the same as y and z in the Cartesian system of Appendix A.

Equations (D-8), (D-9), (D-10), and (D-11) may be incorporated into (D-1) and (D-2).

$$x = a(\cos \beta - e) \tag{D-12}$$

$$y = a(1 - e^2)^{\frac{1}{2}} \sin \beta \tag{D-13}$$

When Eqs. (D-12) and (D-13) are compared with Eqs. (B-50) and (B-51), it is apparent that the coordinate β is equal to the eccentric anomaly E.

Thus, the elliptical cylindrical coordinates α and β for an elliptical path are given by

$$\alpha = \tanh^{-1} (1 - e^2)^{\frac{1}{2}} \tag{D-14}$$

$$\beta = E \tag{D-15}$$

The advantage of this coordinate system lies in the fact that, of the three coordinates α, β, and z, only β is a variable when the path is an ellipse with axes in the directions of x and y.

D.3 Coordinate Curves and Tangent Vectors

The curve obtained by holding two of the three coordinates in a curvilinear system fixed and varying the third is called the coordinate curve of the third coordinate. A tangent vector is defined as a vector tangent to a coordinate curve at a given point and positive in the direction in which the value of the varying coordinate is increasing.

In this section, it will be shown that the tangent vectors of the elliptical cylindrical coordinate system are parallel to the axes of the flight path coordinate system.

The tangent vectors in the α, β, z system are designated $\underline{w}_\alpha$, $\underline{w}_\beta$, $\underline{w}_z$, respectively. The corresponding unit vectors are $\underline{u}_\alpha$, $\underline{u}_\beta$, and $\underline{u}_z$. Similarly, $\underline{u}_x$ and $\underline{u}_y$ are unit vectors in the x and y directions.

The radius vector $\underline{r}$ may be written as

$$\underline{r} = x\,\underline{u}_x + y\,\underline{u}_y + z\,\underline{u}_z \tag{D-16}$$

$$\begin{aligned} &= a e\,(\cosh \alpha \cos \beta - 1)\,\underline{u}_x \\ &\quad + a e \sinh \alpha \sin \beta\,\underline{u}_y + z\,\underline{u}_z \end{aligned} \tag{D-17}$$

The three tangent vectors are

$$\underline{w}_\alpha = \frac{\partial \underline{r}}{\partial \alpha} = a e \sinh \alpha \cos \beta \, \underline{u}_x + a e \cosh \alpha \sin \beta \, \underline{u}_y$$

$$= a \left(1 - e^2\right)^{\frac{1}{2}} \cos E \, \underline{u}_x + a \sin E \, \underline{u}_y \qquad \text{(D-18)}$$

$$\underline{w}_\beta = \frac{\partial \underline{r}}{\partial \beta} = - a e \cosh \alpha \sin \beta \, \underline{u}_x + a e \sinh \alpha \cos \beta \, \underline{u}_y$$

$$= - a \sin E \, \underline{u}_x + a \left(1 - e^2\right)^{\frac{1}{2}} \cos E \, \underline{u}_y \qquad \text{(D-19)}$$

$$\underline{w}_z = \frac{\partial \underline{r}}{\partial z} = \underline{u}_z \qquad \text{(D-20)}$$

The magnitudes of the tangent vectors are

$$|\underline{w}_\alpha| = |\underline{w}_\beta| = \left[a^2 \sin^2 E + a^2 \left(1 - e^2\right) \cos^2 E \right]^{\frac{1}{2}}$$

$$= a \left(1 - e^2 \cos^2 E\right)^{\frac{1}{2}} \qquad \text{(D-21)}$$

$$|\underline{w}_z| = 1 \qquad \text{(D-22)}$$

The unit vectors are

$$\underline{u}_\alpha = \frac{\left(1 - e^2\right)^{\frac{1}{2}} \cos E}{\left(1 - e^2 \cos^2 E\right)^{\frac{1}{2}}} \underline{u}_x + \frac{\sin E}{\left(1 - e^2 \cos^2 E\right)^{\frac{1}{2}}} \underline{u}_y \qquad \text{(D-23)}$$

$$\underline{u}_\beta = - \frac{\sin E}{(1 - e^2 \cos^2 E)^{\frac{1}{2}}} \underline{u}_x + \frac{(1 - e^2)^{\frac{1}{2}} \cos E}{(1 - e^2 \cos^2 E)^{\frac{1}{2}}} \underline{u}_y \qquad \text{(D-24)}$$

$$\underline{u}_z = \underline{u}_z \qquad \text{(D-25)}$$

It may easily be verified that

$$\underline{u}_\alpha \cdot \underline{u}_\beta = \underline{u}_\alpha \cdot \underline{u}_z = \underline{u}_\beta \cdot \underline{u}_z = 0 \qquad \text{(D-26)}$$

and therefore the elliptical cylindrical coordinate system is an orthogonal system.

The orientation of the $\underline{u}_\alpha$ and $\underline{u}_\beta$ vectors with respect to the Cartesian axes x and y may be obtained by forming the dot products of $\underline{u}_\alpha$ with $\underline{u}_x$ and $\underline{u}_y$.

$$\cos(\alpha, x) = \underline{u}_\alpha \cdot \underline{u}_x = \frac{(1 - e^2)^{\frac{1}{2}} \cos E}{(1 - e^2 \cos^2 E)^{\frac{1}{2}}} \qquad \text{(D-27)}$$

$$\cos(\alpha, y) = \sin(\alpha, x) = \underline{u}_\alpha \cdot \underline{u}_y = \frac{\sin E}{(1 - e^2 \cos^2 E)^{\frac{1}{2}}} \qquad \text{(D-28)}$$

where (α, x) is the angle between $\underline{u}_\alpha$ and $\underline{u}_x$, and (α, y) is the angle between $\underline{u}_\alpha$ and $\underline{u}_y$.

Comparison of (D-27) and (D-28) with (B-78) and (B-77), respectively, indicates that the angle between $\underline{u}_\alpha$ and the x-axis is equal to g, the angle between the p-axis of the flight path system and the x-axis. Therefore, $\underline{u}_\alpha$ is parallel to the p-axis. Similarly, $\underline{u}_\beta$ is parallel to the q-axis, and $\underline{u}_z$ is parallel to the z-axis. Thus, it has been proved that the tangent vectors of the α, β, z system are parallel to the axes of the p, q, z system.

This result may be verified by the following deductive process. The β coordinate curve for an elliptical path is obtained by varying E with e and z held constant. This curve is the ellipse itself, and its tangent vector at any point is tangent to the ellipse at that point. The q-axis of the flight path system was chosen to be parallel to the instantaneous orbital velocity vector, and this vector is also tangent to the ellipse at any given point. Therefore, $\underline{u}_\beta$ must be parallel to the q-axis. Moreover, $\underline{u}_z$ is obviously parallel to the z-axis. Since $\underline{u}_\beta$ is parallel to the q-axis, and $\underline{u}_z$ is parallel to the z-axis, and both the elliptical cylindrical coordinate system and the flight path coordinate system are orthogonal systems, it follows that $\underline{u}_\alpha$ must be parallel to the p-axis.

These results may be summarized mathematically in the following two equations:

$$\underline{u}_\alpha = \underline{u}_p \tag{D-29}$$

$$\underline{u}_\beta = \underline{u}_q \tag{D-30}$$

where $\underline{u}_p$ and $\underline{u}_q$ are unit vectors along the p and q axes, respectively.

D.4 Evaluation of the Elliptical Cylindrical Coordinate System

In the elliptical cylindrical coordinate system, the differential change in the radius vector along a known trajectory is

$$d\,\underline{r} = \underline{w}_{\alpha}\, d\,\alpha + \underline{w}_{\beta}\, d\,\beta + \underline{w}_{z}\, d\,z \qquad \text{(D-31)}$$

When the trajectory is an ellipse,

$$d\,\alpha = 0 = d\,z \qquad \text{(D-32)}$$

Then,

$$d\,\underline{r} = \underline{w}_{\beta}\, d\,\beta = a\,(1 - e^2 \cos^2 E)^{\frac{1}{2}}\, d\,E\, \underline{u}_{\beta} \qquad \text{(D-33)}$$

The deviative of $\underline{r}$ with respect to t is

$$\frac{d\,\underline{r}}{dt} = a\,(1 - e^2 \cos^2 E)^{\frac{1}{2}}\, \dot{E}\ \underline{u}_{\beta}$$

$$= \frac{n\,a\,(1 + e \cos E)^{\frac{1}{2}}}{(1 - e \cos E)^{\frac{1}{2}}}\ \underline{u}_{\beta} = \underline{v} \qquad \text{(D-34)}$$

Equation (D-34) represents a simpler, more elegant method of deriving the velocity along an elliptical trajectory than any obtainable by the use of rectilinear coordinate systems. It is in studies of this nature, involving the dynamic or geometric characteristics associated with a known ellipse, that the elliptical cylindrical coordinate system shows to good advantage.

The guidance problem is primarily concerned not with the differential $d\,\underline{r}$, but rather with the variation $\delta\,\underline{r}$. It is important to distinguish between these two quantities. The differential $d\,\underline{r}$ is the infinitesimal change in the position vector $\underline{r}$ due to an infinitesimal displacement along a known reference trajectory. The variation $\delta\,\underline{r}$ is the small difference between the radius vector for an actual trajectory at a given time and the radius vector for a known reference trajectory at the same time.

When an elliptical cylindrical coordinate system is used in conjunction with an elliptical reference trajectory, the differentials of a, e, α, and z are all zero. However, the variations of a, e, α, and z need not be zero, and in general they are not. Thus, the main advantage of the elliptical system, the fact that $d\,\alpha = 0$, is of no consequence when the problem being studied is a variational problem.

The first variation of $\underline{r}$ is

$$\begin{aligned}\delta\,\underline{r} &= \delta x\,\underline{u}_x + \delta y\,\underline{u}_y + \delta z\,\underline{u}_z \\ &= \delta\,(ae \cosh\alpha \cos\beta - 1)\,\underline{u}_x \\ &\quad + \delta\,(ae \sinh\alpha \sin\beta)\,\underline{u}_y + \delta z\,\underline{u}_z \end{aligned} \tag{D-35}$$

$$\begin{aligned} &= \frac{\partial \underline{r}}{\partial a}\,\delta a + \frac{\partial \underline{r}}{\partial e}\,\delta e + \frac{\partial \underline{r}}{\partial \alpha}\,\delta\alpha \\ &\quad + \frac{\partial \underline{r}}{\partial \beta}\,\delta\beta + \frac{\partial \underline{r}}{\partial z}\,\delta z \end{aligned} \tag{D-36}$$

This formulation for $\delta\,\underline{r}$ offers no advantage over that which can be obtained from any of the three reference trajectory rectilinear coordinate systems of Appendix A. Since the coordinate variables in the rectilinear systems are more familiar than those of the elliptical system, no further use will be made of the elliptical system in this analysis.

APPENDIX E
VARIANT EQUATIONS OF MOTION

E.1 Summary

The variant equations of motion of a vehicle in an n-body gravitational field are developed first in vector form and then in component form for the three different reference trajectory coordinate systems. A simplified matrix notation is introduced which indicates that the variation in acceleration is related to the variation in position by means of a symmetric 3-by-3 matrix.

E.2 The Variant Equation in Vector Form

The vector form of the variant equation of motion is obtained from Eq. (B-3) by taking the first variation with respect to $\underline{r}$ at a fixed time. On the left side of (B-3),

$$\delta \left(\frac{\mu}{r^3}\, \underline{r}\right) = \frac{\mu}{r^4}\left(-3\, \underline{r}\, \delta r + r\, \delta \underline{r}\right) \tag{E-1}$$

where the symbol δ signifies the first variation.

On the right side of (B-3),

$$\delta \left(-\frac{G\, m_i}{d_i^3}\, \underline{d}_i\right) = -\frac{G\, m_i}{d_i^4}\left(-3\, \underline{d}_i\, \delta\, d_i + d_i\, \delta \underline{d}_i\right) \tag{E-2}$$

From Fig. B.1,

$$\underline{d}_i = \underline{r} - \underline{r}_i \tag{E-3}$$

Since $\underline{r}_i$ is unaffected by a variation in $\underline{r}$,

$$\delta \underline{d}_i = \delta \underline{r} \tag{E-4}$$

Then,

$$\delta \left(- \frac{G\, m_i}{d_i^3} \underline{d}_i\right) = \frac{G\, m_i}{d_i^4} \left(3\, \underline{d}_i\, \delta d_i - d_i\, \delta \underline{r}\right) \tag{E-5}$$

The variation in the last term of (B-3) due to $\delta \underline{r}$ is zero.

The variant equation in vector form is

$$\delta \ddot{\underline{r}} = \frac{\mu}{r^4} \left(3\, \underline{r}\, \delta r - r\, \delta \underline{r}\right) + G \sum_{i=1}^{n} \frac{m_i}{d_i^4} \left(3\, \underline{d}_i\, \delta d_i - d_i\, \delta \underline{r}\right) \tag{E-6}$$

E. 3 Variant Equations in the Reference Trajectory Coordinate Systems

Since the x, y, z coordinate system is non-rotating, $\delta \underline{r}$ and $\delta \ddot{\underline{r}}$ are obtained directly from (B-4) and (B-6).

$$\delta \underline{r} = \delta x\, \underline{u}_x + \delta y\, \underline{u}_y + \delta z\, \underline{u}_z \tag{E-7}$$

$$\delta \ddot{\underline{r}} = \delta \ddot{x}\, \underline{u}_x + \delta \ddot{y}\, \underline{u}_y + \delta \ddot{z}\, \underline{u}_z \tag{E-8}$$

δr and δd_i are derived from (B-7) and (B-8).

$$\delta r = \frac{x}{r}\, \delta x + \frac{y}{r}\, \delta y + \frac{z}{r}\, \delta z \tag{E-9}$$

$$\delta d_i = \frac{(x - x_i)}{d_i}\, \delta x + \frac{(y - y_i)}{d_i}\, \delta y + \frac{(z - z_i)}{d_i}\, \delta z \tag{E-10}$$

Equations (E-7) through (E-10) are substituted into (E-6), and the resulting equation is written in matrix form.

$$\begin{pmatrix} \delta\ddot{x} \\ \delta\ddot{y} \\ \delta\ddot{z} \end{pmatrix} = \left\{ \frac{\mu}{r^5} \left[3 \begin{pmatrix} x^2 & xy & xz \\ yx & y^2 & yz \\ zx & zy & z^2 \end{pmatrix} - r^2 \overset{*}{I}_3 \right] \right.$$

$$\left. + G \sum_{i=1}^{n} \frac{m_i}{d_i^5} \left[3 \begin{pmatrix} (x-x_i)^2 & (x-x_i)(y-y_i) & (x-x_i)(z-z_i) \\ (y-y_i)(x-x_i) & (y-y_i)^2 & (y-y_i)(z-z_i) \\ (z-z_i)(x-x_i) & (z-z_i)(y-y_i) & (z-z_i)^2 \end{pmatrix} - d_i^2 \overset{*}{I}_3 \right] \right\} \begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix} \tag{E-11}$$

where $\overset{*}{I}_3$ is the 3-by-3 identity matrix. An asterisk above a capital letter indicates that the letter represents a matrix.

Equation (E-11) may be written more compactly as follows:

$$\delta\ddot{\underline{r}} = \left[\frac{\mu}{r^5} \left(3\, \underline{r}\, \underline{r}^T - \underline{r}^T \underline{r}\, \overset{*}{I}_3\right) + G \sum_{i=1}^{n} \frac{m_i}{d_i^5} \left(3\, \underline{d}_i\, \underline{d}_i^T - \underline{d}_i^T\, \underline{d}_i\, \overset{*}{I}_3\right) \right] \delta\underline{r} \tag{E-12}$$

where $\underline{r}$, $\underline{d}_i$, and $\delta\underline{r}$ are three-dimensional column vectors and the superscript T indicates the transpose.

The expression inside the square brackets in (E-12) is a symmetrical 3-by-3 matrix which is designated $\overset{*}{G}$.

$$\delta\ddot{\underline{r}} = \overset{*}{G}\ \delta\underline{r} \tag{E-13}$$

In the r, s, z coordinate system, rotating with angular velocity $\dot{f}$,

$$\underline{r} + \delta\underline{r} = (\rho + \delta\rho)\,\underline{u}_r + \delta s\,\underline{u}_s + (z + \delta z)\,\underline{u}_z \tag{E-14}$$

$$\begin{aligned}\underline{v} + \delta\underline{v} &= (\dot{\rho} + \delta\dot{\rho} - \dot{f}\,\delta s)\,\underline{u}_r \\ &\quad + (\delta\dot{s} + \rho\,\dot{f} + \dot{f}\,\delta\rho)\,\underline{u}_s + (\dot{z} + \delta\dot{z})\,\underline{u}_z\end{aligned} \tag{E-15}$$

$$\begin{aligned}\underline{a} + \delta\underline{a} &= (\ddot{\rho} - \rho\,\dot{f}^2 + \delta\ddot{\rho} - \dot{f}^2\,\delta\rho - 2\dot{f}\,\delta\dot{s} \\ &\quad - \ddot{f}\,\delta s)\,\underline{u}_r + (\rho\,\ddot{f} + 2\,\dot{\rho}\,\dot{f} + 2\dot{f}\,\delta\dot{\rho} + \ddot{f}\,\delta\rho \\ &\quad + \delta\ddot{s} - \dot{f}^2\,\delta s)\,\underline{u}_s + (\ddot{z} + \delta\ddot{z})\,\underline{u}_z\end{aligned} \tag{E-16}$$

$\delta\underline{a}$ is obtained by subtracting (B-13) from (E-16).

$$\begin{aligned}\delta\underline{a} = \delta\ddot{\underline{r}} &= (\delta\ddot{\rho} - \dot{f}^2\,\delta\rho - 2\dot{f}\,\delta\dot{s} - \ddot{f}\,\delta s)\,\underline{u}_r \\ &\quad + (2\dot{f}\,\delta\dot{\rho} + \ddot{f}\,\delta\rho + \delta\ddot{s} - \dot{f}^2\,\delta s)\,\underline{u}_s + \delta\ddot{z}\,\underline{u}_z\end{aligned} \tag{E-17}$$

Then Eq. (E-12) may be written as

$$\begin{aligned}\begin{pmatrix} \delta\ddot{\rho} - \dot{f}^2\,\delta\rho - 2\dot{f}\,\delta\dot{s} - \ddot{f}\,\delta s \\ 2\dot{f}\,\delta\dot{\rho} + \ddot{f}\,\delta\rho + \delta\ddot{s} - \dot{f}^2\,\delta s \\ \delta\ddot{z} \end{pmatrix} &= \Bigg\{ \frac{\mu}{r^5}\left[3\begin{pmatrix} \rho^2 & 0 & \rho z \\ 0 & 0 & 0 \\ z\rho & 0 & z^2 \end{pmatrix} - r^2 I_3^*\right] \\ &\quad + G\sum_{i=1}^{n}\frac{m_i}{d_i^5}\left[3\begin{pmatrix} (\rho - \rho_i)^2 & -(\rho - \rho_i)\,s_i & (\rho - \rho_i)(z - z_i) \\ -s_i(\rho - \rho_i) & s_i^2 & -s_i(z - z_i) \\ (z - z_i)(\rho - \rho_i) & -(z - z_i)\,s_i & (z - z_i)^2 \end{pmatrix} - d_i^2 I_3^*\right]\Bigg\}\begin{pmatrix} \delta\rho \\ \delta s \\ \delta z \end{pmatrix}\end{aligned} \tag{E-18}$$

A similar development can be carried out in the p, q, z coordinate system, which rotates with angular velocity $\dot{g}$. The resulting equation is

$$\begin{pmatrix} \delta\ddot{p} - \dot{g}^2\,\delta p - 2\dot{g}\,\delta\dot{q} - \ddot{g}\,\delta q \\ 2\dot{g}\,\delta\dot{p} + \ddot{g}\,\delta p + \delta\ddot{q} - \dot{g}^2\,\delta q \\ \delta\ddot{z} \end{pmatrix} = \left\{ \frac{\mu}{r^5}\left[3\begin{pmatrix} p^2 & pq & pz \\ qp & q^2 & qz \\ zp & zq & z^2 \end{pmatrix} - r^2\,\overset{*}{I}_3 \right] \right.$$

$$\left. + G\sum_{i=1}^{n} \frac{m_i}{d_i^5}\left[3\begin{pmatrix} (p-p_i)^2 & (p-p_i)(q-q_i) & (p-p_i)(z-z_i) \\ (q-q_i)(p-p_i) & (q-q_i)^2 & (q-q_i)(z-z_i) \\ (z-z_i)(p-p_i) & (z-z_i)(q-q_i) & (z-z_i)^2 \end{pmatrix} - d_i^2\,\overset{*}{I}_3 \right] \right\} \begin{pmatrix} \delta p \\ \delta q \\ \delta z \end{pmatrix} \tag{E-19}$$

E.4 Symmetry of Matrix $\overset{*}{G}$

An interesting physical explanation of the symmetry of the matrix $\overset{*}{G}$ in Eq. (E-13) has been given by McLean, Schmidt, and McGee.*

Since all the forces being considered are gravitational, a scalar potential V may be defined such that

$$\underline{\ddot{r}} = \nabla\ V \tag{E-20}$$

where ∇ signifies the gradient of a scalar quantity. In matrix form in the x y z coordinate system,

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial V}{\partial x} \\ \dfrac{\partial V}{\partial y} \\ \dfrac{\partial V}{\partial z} \end{pmatrix} \tag{E-21}$$

---

* Page 34 of Reference (13).

The variation in $\ddot{x}$ is

$$\delta \ddot{x} = \frac{\partial \ddot{x}}{\partial x} \delta x + \frac{\partial \ddot{x}}{\partial y} \delta y + \frac{\partial \ddot{x}}{\partial z} \delta z$$

$$= \frac{\partial^2 V}{\partial x^2} \delta x + \frac{\partial^2 V}{\partial y \, \partial x} \delta y + \frac{\partial^2 V}{\partial z \, \partial x} \delta z \qquad \text{(E-22)}$$

Analogous equations may be written for $\delta \ddot{y}$ and $\delta \ddot{z}$. Then the vector $\delta \underline{\ddot{r}}$ is given by

$$\delta \underline{\ddot{r}} = \begin{pmatrix} \delta \ddot{x} \\ \delta \ddot{y} \\ \delta \ddot{z} \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 V}{\partial x^2} & \frac{\partial^2 V}{\partial y \, \partial x} & \frac{\partial^2 V}{\partial z \, \partial x} \\ \frac{\partial^2 V}{\partial x \, \partial y} & \frac{\partial^2 V}{\partial y^2} & \frac{\partial^2 V}{\partial z \, \partial y} \\ \frac{\partial^2 V}{\partial x \, \partial z} & \frac{\partial^2 V}{\partial y \, \partial z} & \frac{\partial^2 V}{\partial z^2} \end{pmatrix} \begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix} \qquad \text{(E-23)}$$

A comparison of (E-23) with (E-13) indicates that the 3-by-3 matrix in (E-23) is $\overset{*}{G}$ and that its symmetry is due to the fact that

$$\frac{\partial^2 V}{\partial r_i \, \partial r_j} = \frac{\partial^2 V}{\partial r_j \, \partial r_i}; \quad i, j, = 1, 2, 3 \qquad \text{(E-24)}$$

where r_i and r_j are components of $\underline{r}$.

Thus, as long as the force field is conservative, the matrix $\overset{*}{G}$ relating $\delta \underline{\ddot{r}}$ to $\delta \underline{r}$ is symmetric. The inclusion of the effects of earth oblateness in the analysis does not affect the symmetry of $\overset{*}{G}$.

APPENDIX F
GENERAL MATRIX FORMULATIONS

F.1 Summary

Several different types of matrix formulations are introduced to represent the solution of the variant equations of motion. The interrelationships among the various matrices are developed. A method is indicated for evaluating the terms in the matrices by the use of numerical integration. Some interesting symmetry properties of the matrices are proved. The symmetry properties are used to find the inverse of the basic 6-by-6 matrix by inspection.

F.2 Path Deviation

Just as the solution of the general equations of motion involves six constants, so does the solution of the variant equations of motion. The analogy may be carried further. It was pointed out in Section B.6 that the six constants in the general solution may be the three components of position and the three components of velocity occurring at a specified time; in the variant solution the constants may be the variations in the three components of position and the three components of velocity occurring at a specified time. The constants in the variant solution may also be the variations in the components of position at two different specified times. If the motion is two-body motion, variations in the six orbital elements may be used. Any one of these groupings of six constants may be regarded as a six-component vector. This type of vector will be referred to as the path deviation vector.

The mathematical representations for the three classes of path deviation vectors mentioned above are

(1) $$\begin{Bmatrix} \delta \underline{r}_k \\ \delta \underline{v}_k \end{Bmatrix}$$

(2) $$\begin{Bmatrix} \delta \underline{r}_i \\ \delta \underline{r}_j \end{Bmatrix}$$

(3) $$\delta \underline{e}$$

$\delta \underline{r}_k$ and $\delta \underline{v}_k$ are, respectively, the position variation and the velocity variation at time t_k. They may be grouped together into a single vector, which will be designated $\delta \underline{x}_k$. $\delta \underline{r}_i$ and $\delta \underline{r}_j$ are the position variations at times t_i and t_j. $\delta \underline{e}$ consists of the variations in some grouping of six orbital elements.

The three different path deviation vectors may be related to each other as follows:

$$\delta \underline{e} = \{ \overset{*}{R}_k \quad \overset{*}{V}_k \} \begin{Bmatrix} \delta \underline{r}_k \\ \delta \underline{v}_k \end{Bmatrix} = \{ \overset{*}{R}_k \quad \overset{*}{V}_k \} \; \delta \underline{x}_k \tag{F-1}$$

$$= \overset{*}{H}_{ij} \, \delta \underline{r}_i + \overset{*}{H}_{ji} \, \delta \underline{r}_j \tag{F-2}$$

where

$$\overset{*}{R}_k = \left\{ \frac{\partial \underline{e}}{\partial \underline{r}_k} \Big|_{\delta \underline{v}_k = \text{constant}} \right\} \tag{F-3}$$

$$\overset{*}{V}_k = \left\{ \frac{\partial \underline{e}}{\partial \underline{v}_k} \Big|_{\delta \underline{r}_k = \text{constant}} \right\} \tag{F-4}$$

$$\overset{*}{H}_{ij} = \left\{ \frac{\partial \underline{e}}{\partial \underline{r}_i} \Big|_{\delta \underline{r}_j = \text{constant}} \right\} \tag{F-5}$$

$\overset{*}{R}_k$, $\overset{*}{V}_k$, $\overset{*}{H}_{ij}$ and $\overset{*}{H}_{ji}$ are all 6-by-3 matrices. The subscript k in $\overset{*}{R}_k$ and $\overset{*}{V}_k$ indicates that the elements of the two matrices are functions of t_k. Similarly, the elements of $\overset{*}{H}_{ij}$ and $\overset{*}{H}_{ji}$ are functions of t_i and t_j.

F. 3 Variation in Position

The variation in position at any arbitrary time t_m may be expressed in terms of the path deviation vector.

$$\delta \underline{r}_m = \overset{*}{F}_m \, \delta \underline{e} \tag{F-6}$$

$$= \overset{*}{F}_m \{\overset{*}{R}_k \quad \overset{*}{V}_k\} \, \delta \underline{x}_k \tag{F-7}$$

$$= \overset{*}{F}_m \{\overset{*}{H}_{ij} \, \delta \underline{r}_i + \overset{*}{H}_{ji} \, \delta \underline{r}_j\} \tag{F-8}$$

$$= \{\overset{*}{M}_{mk} \quad \overset{*}{N}_{mk}\} \, \delta \underline{x}_k \tag{F-9}$$

where

$$\overset{*}{F}_m = \left\{ \frac{\partial \underline{r}_m}{\partial \underline{e}} \right\} \tag{F-10}$$

$$\overset{*}{M}_{mk} = \left\{ \frac{\partial \underline{r}_m}{\partial \underline{r}_k} \Bigg|_{\delta \underline{r}_k = \text{ constant}} \right\} \tag{F-11}$$

$$\overset{*}{N}_{mk} = \left\{ \frac{\partial \underline{r}_m}{\partial \underline{r}_k} \Bigg|_{\delta \underline{v}_k = \text{ constant}} \right\} \tag{F-12}$$

$\overset{*}{F}_m$ is a 3-by-6 matrix. $\overset{*}{M}_{mk}$ and $\overset{*}{N}_{mk}$ are 3-by-3 matrices.

Since the elements of the path deviation vector are independent of each other, it is apparent from (F-7), (F-8), and (F-9) that

$$\overset{*}{F}_k \overset{*}{R}_k = \overset{*}{F}_i \overset{*}{H}_{ij} = \overset{*}{M}_{kk} = \overset{*}{I}_3 \tag{F-13}$$

$$\overset{*}{F}_k \overset{*}{V}_k = \overset{*}{F}_i \overset{*}{H}_{ji} = \overset{*}{N}_{kk} = \overset{*}{O}_3 \tag{F-14}$$

where $\overset{*}{O}_3$ is the 3-by-3 zero matrix.

In general,

$$\overset{*}{M}_{mk} = \overset{*}{F}_m \overset{*}{R}_k \tag{F-15}$$

$$\overset{*}{N}_{mk} = \overset{*}{F}_m \overset{*}{V}_k \tag{F-16}$$

Equation (F-15) indicates that $\overset{*}{M}_{mk}$, whose elements are functions of both t_m and t_k, can be written as the product of two matrices, the elements of one being functions solely of t_m and the elements of the second being functions solely of t_k A similar statement may be made with respect to $\overset{*}{N}_{mk}$.

F.4 Variation in Velocity

The variation in velocity at time t_m is

$$\delta \underline{v}_m = \overset{*}{L}_m \, \delta \underline{e} \tag{F-17}$$

$$= \overset{*}{L}_m \left\{ \overset{*}{R}_k \quad \overset{*}{V}_k \right\} \delta \underline{x}_k \tag{F-18}$$

$$= \overset{*}{L}_m \left\{ \overset{*}{H}_{ij} \, \delta \underline{r}_i + \overset{*}{H}_{ji} \, \delta \underline{r}_j \right\} \tag{F-19}$$

$$= \left\{ \overset{*}{S}_{mk} \quad \overset{*}{T}_{mk} \right\} \delta \underline{x}_k \tag{F-20}$$

where

$$\overset{*}{L}_m = \left\{ \frac{\partial \underline{v}_m}{\partial \underline{e}} \right\} \tag{F-21}$$

$$\overset{*}{S}_{mk} = \left\{ \frac{\partial \underline{v}_m}{\partial \underline{r}_k} \Bigg|_{\delta \underline{v}_k = \text{constant}} \right\} \tag{F-22}$$

$$\overset{*}{T}_{mk} = \left\{ \frac{\partial \underline{v}_m}{\partial \underline{v}_k} \Bigg|_{\delta \underline{r}_k = \text{constant}} \right\} \tag{F-23}$$

From (F-18) and (F-20),

$$\overset{*}{S}_{mk} = \overset{*}{L}_m \overset{*}{R}_k \tag{F-24}$$

$$\overset{*}{T}_{mk} = \overset{*}{L}_m \overset{*}{V}_k \tag{F-25}$$

The equations corresponding to (F-13) and (F-14) are

$$\overset{*}{T}_{kk} = \overset{*}{L}_k \overset{*}{V}_k = \overset{*}{I}_3 \tag{F-26}$$

$$\overset{*}{S}_{kk} = \overset{*}{L}_k \overset{*}{R}_k = \overset{*}{O}_3 \tag{F-27}$$

When $t_m = t_i$, Eq. (F-19) becomes

$$\delta \underline{v}_i = \overset{*}{L}_i \left\{ \overset{*}{H}_{ij}\, \delta \underline{r}_i + \overset{*}{H}_{ji}\, \delta \underline{r}_j \right\} \tag{F-28}$$

$$= \overset{*}{J}_{ij}\, \delta \underline{r}_i + \overset{*}{K}_{ij}\, \delta \underline{r}_j \tag{F-29}$$

where

$$\overset{*}{J}_{ij} = \overset{*}{L}_i \overset{*}{H}_{ij} = \left\{ \frac{\partial \underline{v}_i}{\partial \underline{r}_i} \bigg|_{\delta \underline{r}_j = \text{constant}} \right\} \tag{F-30}$$

$$\overset{*}{K}_{ij} = \overset{*}{L}_i \overset{*}{H}_{ji} = \left\{ \frac{\partial \underline{v}_i}{\partial \underline{r}_j} \bigg|_{\delta \underline{r}_i = \text{constant}} \right\} \tag{F-31}$$

$\overset{*}{J}_{ij}$ and $\overset{*}{K}_{ij}$ are 3-by-3 matrices.

(F-29) may be solved for $\delta \underline{r}_j$ by pre-multiplying the terms of the equation by $\overset{*}{K}_{ij}^{-1}$, where the superscript -1 indicates the inverse of a square matrix.

$$\delta \underline{r}_j = - \overset{*}{K}_{ij}^{-1} \overset{*}{J}_{ij} \, \delta \underline{r}_i + \overset{*}{K}_{ij}^{-1} \, \delta \underline{v}_i$$

$$= \overset{*}{K}_{ij}^{-1} \{ - \overset{*}{J}_{ij} \quad \overset{*}{I}_3 \} \; \delta \underline{x}_i \tag{F-32}$$

Comparison of (F-32) with (F-9) indicates that

$$\overset{*}{M}_{ji} = - \overset{*}{K}_{ij}^{-1} \overset{*}{J}_{ij} \tag{F-33}$$

$$\overset{*}{N}_{ji} = \overset{*}{K}_{ij}^{-1} \tag{F-34}$$

The path deviation vector at time t_j may be expressed in terms of the path deviation vector at time t_i as follows:

$$\delta \underline{x}_j = \begin{Bmatrix} \delta \underline{r}_j \\ \delta \underline{v}_j \end{Bmatrix} = \begin{Bmatrix} \overset{*}{M}_{ji} & \overset{*}{N}_{ji} \\ \overset{*}{S}_{ji} & \overset{*}{T}_{ji} \end{Bmatrix} \delta \underline{x}_i = \overset{*}{C}_{ji} \, \delta \underline{x}_i \tag{F-35}$$

where

$$\overset{*}{C}_{ji} = \begin{Bmatrix} \overset{*}{M}_{ji} & \overset{*}{N}_{ji} \\ \overset{*}{S}_{ji} & \overset{*}{T}_{ji} \end{Bmatrix} = \left\{ \frac{\partial \underline{x}_j}{\partial \underline{x}_i} \right\} \tag{F-36}$$

The 6-by-6 matrix $\overset{*}{C}_{ji}$ is known as the transition matrix. It follows from (F-35) that

$$\delta \underline{x}_j = \overset{*}{C}_{ji} \overset{*}{C}_{ij} \delta \underline{x}_j \qquad \text{(F-37)}$$

$$\therefore \quad \overset{*}{C}_{ji} \overset{*}{C}_{ij} = \overset{*}{I}_6 \qquad \text{(F-38)}$$

where $\overset{*}{I}_6$ is the 6-by-6 identity matrix. Then,

$$\overset{*}{M}_{ji} \overset{*}{M}_{ij} + \overset{*}{N}_{ji} \overset{*}{S}_{ij} = \overset{*}{I}_3 = \overset{*}{S}_{ji} \overset{*}{N}_{ij} + \overset{*}{T}_{ji} \overset{*}{T}_{ij} \qquad \text{(F-39)}$$

$$\overset{*}{M}_{ji} \overset{*}{N}_{ij} + \overset{*}{N}_{ji} \overset{*}{T}_{ij} = \overset{*}{O}_3 = \overset{*}{S}_{ji} \overset{*}{M}_{ij} + \overset{*}{T}_{ji} \overset{*}{S}_{ij} \qquad \text{(F-40)}$$

F.5 Matrix Differential Equations

The position of a point P with respect to the origin of a rotating coordinate system may be represented by the vector $\underline{r}$, the components of $\underline{r}$ in the rotating system being r_1, r_2, and r_3. The angular velocity of the system with respect to inertial space is $\underline{\omega}$, with components ω_1, ω_2, and ω_3.

The velocity of P in a non-rotating coordinate system is related to its velocity in the rotating system by the equation

$$\underline{v} = \left(\frac{d\underline{r}}{dt}\right)_{NR} = \left(\frac{d\underline{r}}{dt}\right)_{R} + \underline{\omega} \times \underline{r} \qquad \text{(F-41)}$$

where the subscripts NR and R refer, respectively, to the non-rotating and rotating coordinate systems. The matrix form of Eq. (F-41) is

$$\begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} \dot{r}_1 + \omega_2 r_3 - \omega_3 r_2 \\ \dot{r}_2 + \omega_3 r_1 - \omega_1 r_3 \\ \dot{r}_3 + \omega_1 r_2 - \omega_2 r_1 \end{Bmatrix} \tag{F-42}$$

$$= \begin{Bmatrix} \dot{r}_1 \\ \dot{r}_2 \\ \dot{r}_3 \end{Bmatrix} + \overset{*}{W} \underline{r} \tag{F-43}$$

where $\overset{*}{W}$ is given by

$$\overset{*}{W} = \begin{Bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{Bmatrix} \tag{F-44}$$

$\overset{*}{W}$ is a skew-symmetric matrix, i.e.,

$$\overset{*}{W}^T = - \overset{*}{W} \tag{F-45}$$

The variation of $\underline{v}$ is

$$\delta \underline{v} = \delta \left(\frac{d\,\underline{r}}{dt}\right)_{NR} = \delta \left(\frac{d\,\underline{r}}{dt}\right)_R + \left(\delta\, \overset{*}{W}\right) \underline{r} + \overset{*}{W}\, \delta \underline{r} \tag{F-46}$$

Since the angular velocity of the coordinate system is not affected by variations in $\underline{r}$,

$$\delta \overset{*}{W} = \overset{*}{O}_3 \tag{F-47}$$

From Eq. (F-9), the variation of the velocity of P in the rotating system is

$$\left[\frac{d\,(\delta \underline{r}_i)}{d\ t_i}\right]_R = \begin{Bmatrix} \delta \dot{r}_{i1} \\ \delta \dot{r}_{i2} \\ \delta \dot{r}_{i3} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial \overset{*}{M}_{ij}}{\partial t_i} & \dfrac{\partial \overset{*}{N}_{ij}}{\partial t_i} \end{Bmatrix} \delta \underline{x}_j \tag{F-48}$$

Equations (F-47) and (F-48) are substituted into (F-46), and the resulting expression is equated to (F-20).

$$\begin{aligned} \delta \underline{v}_i &= \begin{Bmatrix} \dfrac{\partial \overset{*}{M}_{ij}}{\partial t_i} + \overset{*}{W}_i \overset{*}{M}_{ij} & \dfrac{\partial \overset{*}{N}_{ij}}{\partial t_i} + \overset{*}{W}_i \overset{*}{N}_{ij} \end{Bmatrix} \delta \underline{x}_j \\ &= \begin{Bmatrix} \overset{*}{S}_{ij} & \overset{*}{T}_{ij} \end{Bmatrix} \delta \underline{x}_j \end{aligned} \tag{F-49}$$

In similar fashion $\delta \underline{a}_i$ may be written in terms of $\overset{*}{S}_{ij}$, $\overset{*}{T}_{ij}$, and their derivatives, and then equated to the right-hand side of (E-13).

$$\begin{aligned} \delta \underline{a}_i &= \begin{Bmatrix} \dfrac{\partial \overset{*}{S}_{ij}}{\partial t_i} + \overset{*}{W}_i \overset{*}{S}_{ij} & \dfrac{\partial \overset{*}{T}_{ij}}{\partial t_i} + \overset{*}{W}_i \overset{*}{T}_{ij} \end{Bmatrix} \delta \underline{x}_j \\ &= \overset{*}{G}_i \, \delta \underline{r}_i = \overset{*}{G}_i \begin{Bmatrix} \overset{*}{M}_{ij} & \overset{*}{N}_{ij} \end{Bmatrix} \delta \underline{x}_j \end{aligned} \tag{F-50}$$

By equating coefficients of $\delta \underline{r}_j$ and also of $\delta \underline{v}_j$ in (F-49) and (F-50), the following matrix differential equations are obtained:

$$\frac{\partial \overset{*}{M}_{ij}}{\partial t_i} + \overset{*}{W}_i \overset{*}{M}_{ij} = \overset{*}{S}_{ij} \tag{F-51}$$

$$\frac{\partial \overset{*}{S}_{ij}}{\partial t_i} + \overset{*}{W}_i \overset{*}{S}_{ij} = \overset{*}{G}_i \overset{*}{M}_{ij} \tag{F-52}$$

$$\frac{\partial \overset{*}{N}_{ij}}{\partial t_i} + \overset{*}{W}_i \overset{*}{N}_{ij} = \overset{*}{T}_{ij} \tag{F-53}$$

$$\frac{\partial \overset{*}{T}_{ij}}{\partial t_i} + \overset{*}{W}_i \overset{*}{T}_{ij} = \overset{*}{G}_i \overset{*}{N}_{ij} \tag{F-54}$$

Matrix differential equations may also be obtained for the 3-by-6 matrices $\overset{*}{F}_i$ and $\overset{*}{L}_i$. These are derived from Eqs. (F-6), (F-17), and (E-13).

$$\delta \underline{v}_i = \left\{ \frac{d \overset{*}{F}_i}{dt_i} + \overset{*}{W}_i \overset{*}{F}_i \right\} \delta \underline{e} = \overset{*}{L}_i \, \delta \underline{e} \tag{F-55}$$

$$\delta \underline{a}_i = \left\{ \frac{d \overset{*}{L}_i}{dt_i} + \overset{*}{W}_i \overset{*}{L}_i \right\} \delta \underline{e} = \overset{*}{G}_i \overset{*}{F}_i \, \delta \underline{e} \tag{F-56}$$

The differential equations corresponding to (F-51) through (F-54) are

$$\frac{d \overset{*}{F}_i}{dt_i} + \overset{*}{W}_i \overset{*}{F}_i = \overset{*}{L}_i \tag{F-57}$$

$$\frac{d \overset{*}{L}_i}{dt_i} + \overset{*}{W}_i \overset{*}{L}_i = \overset{*}{G}_i \overset{*}{F}_i \tag{F-58}$$

Finally, it is of interest to relate the 6-by-3 matrices $\overset{*}{R}_i$ and $\overset{*}{V}_i$ to each other. This may be done by taking the time derivative of Eq. (F-1) with respect to the rotating coordinate system and equating the result to $\underline{O}_6$, the six-component zero vector.

$$\frac{d}{dt_i}(\delta \underline{e}) = \frac{d\overset{*}{R}_i}{dt_i}\delta \underline{r}_i + \frac{d\overset{*}{V}_i}{dt_i}\delta \underline{v}_i$$

$$+ \overset{*}{R}_i \frac{d}{dt_i}(\delta \underline{r}_i) + \overset{*}{V}_i \frac{d}{dt_i}(\delta \underline{v}_i) = \underline{O}_6 \qquad \text{(F-59)}$$

The derivatives of the variation vectors with respect to the rotating system are

$$\frac{d}{dt_i}(\delta \underline{r}_i) = \delta \underline{v}_i - \overset{*}{W}_i \delta \underline{r}_i \qquad \text{(F-60)}$$

$$\frac{d}{dt_i}(\delta \underline{v}_i) = \delta \underline{a}_i - \overset{*}{W}_i \delta \underline{v}_i$$

$$= \overset{*}{G}_i \delta \underline{r}_i - \overset{*}{W}_i \delta \underline{v}_i \qquad \text{(F-61)}$$

(F-60) and (F-61) are substituted into (F-59).

$$\left\{\frac{d\overset{*}{R}_i}{dt_i} - \overset{*}{R}_i \overset{*}{W}_i + \overset{*}{V}_i \overset{*}{G}_i\right\}\delta \underline{r}_i$$

$$+ \left\{\frac{d\overset{*}{V}_i}{dt_i} - \overset{*}{V}_i \overset{*}{W}_i + \overset{*}{R}_i\right\}\delta \underline{v}_i = \underline{O}_6 \qquad \text{(F-62)}$$

The coupled matrix differential equations are obtained from the coefficients of $\delta \underline{r}_i$ and $\delta \underline{v}_i$ in (F-62).

$$\frac{d \overset{*}{R}_i}{dt_i} - \overset{*}{R}_i \overset{*}{W}_i = - \overset{*}{V}_i \overset{*}{G}_i \tag{F-63}$$

$$\frac{d \overset{*}{V}_i}{dt_i} - \overset{*}{V}_i \overset{*}{W}_i = - \overset{*}{R}_i \tag{F-64}$$

Equations (F-63) and (F-64) may be used to get relations involving the first partial derivatives of $\overset{*}{M}_{ij}$, $\overset{*}{N}_{ij}$, $\overset{*}{S}_{ij}$, and $\overset{*}{T}_{ij}$ with respect to t_j.

$$\frac{\partial \overset{*}{M}_{ij}}{\partial t_j} = \overset{*}{F}_i \frac{d \overset{*}{R}_j}{dt_j} = \overset{*}{F}_i \left\{ \overset{*}{R}_j \overset{*}{W}_j - \overset{*}{V}_j \overset{*}{G}_j \right\} \tag{F-65}$$

$$\frac{\partial \overset{*}{N}_{ij}}{\partial t_j} = \overset{*}{F}_i \frac{d \overset{*}{V}_j}{dt_j} = \overset{*}{F}_i \left\{ \overset{*}{V}_j \overset{*}{W}_j - \overset{*}{R}_j \right\} \tag{F-66}$$

$$\frac{\partial \overset{*}{S}_{ij}}{\partial t_j} = \overset{*}{L}_i \frac{d \overset{*}{R}_j}{dt_j} = \overset{*}{L}_i \left\{ \overset{*}{R}_j \overset{*}{W}_j - \overset{*}{V}_j \overset{*}{G}_j \right\} \tag{F-67}$$

$$\frac{\partial \overset{*}{T}_{ij}}{dt_j} = \overset{*}{L}_i \frac{d \overset{*}{V}_j}{dt_j} = \overset{*}{L}_i \left\{ \overset{*}{V}_j \overset{*}{W}_j - \overset{*}{R}_j \right\} \tag{F-68}$$

These four equations may be written as matrix differential equations in $\overset{*}{M}_{ij}$, $\overset{*}{N}_{ij}$, $\overset{*}{S}_{ij}$, and $\overset{*}{T}_{ij}$.

$$\frac{\partial \overset{*}{M}_{ij}}{\partial t_j} - \overset{*}{M}_{ij} \overset{*}{W}_j = - \overset{*}{N}_{ij} \overset{*}{G}_j \tag{F-69}$$

$$\frac{\partial \overset{*}{N}_{ij}}{\partial t_j} - \overset{*}{N}_{ij} \overset{*}{W}_j = - \overset{*}{M}_{ij} \tag{F-70}$$

$$\frac{\partial \overset{*}{S}_{ij}}{\partial t_j} - \overset{*}{S}_{ij} \overset{*}{W}_j = - \overset{*}{T}_{ij} \overset{*}{G}_j \tag{F-71}$$

$$\frac{\partial \overset{*}{T}_{ij}}{\partial t_j} - \overset{*}{T}_{ij} \overset{*}{W}_j = - \overset{*}{S}_{ij} \tag{F-72}$$

F. 6 Numerical Integration

The variant equations of motion are represented by (E-13). The solution of these equations for $\delta \underline{r}$ as a function of time is represented by (F-9). The problem now is to evaluate the elements of $\overset{*}{M}_{ij}$ and $\overset{*}{N}_{ij}$.

No direct analytical solution of (E-13) has yet been devised for the case when there are disturbing forces which affect the motion of the vehicle. However, the elements of $\overset{*}{M}_{ij}$ and $\overset{*}{N}_{ij}$ may be found by numerical integration of the coupled equations (F-51) through (F-54).

Since $\overset{*}{M}_{ij}$, $\overset{*}{N}_{ij}$, $\overset{*}{S}_{ij}$, $\overset{*}{T}_{ij}$ are all 3-by-3 matrices, each of the four matrix differential equations represents nine first-order linear differential equations. The elements of the 3-by-3 matrices $\overset{*}{W}_i$ and $\overset{*}{G}_i$ are known functions of the characteristics of the reference trajectory.

Equations (F-51) and (F-52) are coupled equations in the elements of $\overset{*}{M}_{ij}$ and $\overset{*}{S}_{ij}$. They can be integrated numerically if initial values are known for the elements of $\overset{*}{M}_{ij}$ and $\overset{*}{S}_{ij}$. Fortunately, such initial values are available from (F-13) and (F-27).

$$\overset{*}{M}_{jj} = \overset{*}{I}_3 \qquad \overset{*}{S}_{jj} = \overset{*}{O}_3 \tag{F-73}$$

Similarly, (F-53) and (F-54) are coupled equations in $\overset{*}{N}_{ij}$ and $\overset{*}{T}_{ij}$ which can be integrated numerically since the initial values are given by

$$\overset{*}{N}_{jj} = \overset{*}{O}_3 \qquad \overset{*}{T}_{jj} = \overset{*}{I}_3 \tag{F-74}$$

The integrations are carried out at a fixed value of t_j. The independent variable is t_i.

The computation can be simplified if a non-rotating coordinate system is used, for in that case the matrix $\overset{*}{W}_i$ vanishes. The accuracy of the computation is improved if the z-axis of the coordinate system is perpendicular to the plane of the motion that would occur if there were no disturbing forces; i.e., if the coordinate system is one of the reference trajectory systems described in Appendix A. With this choice of coordinates, the motion in the z direction is relatively loosely coupled (through the disturbing forces) to the motion in the plane perpendicular to the z-axis, and consequently four of the nine elements in each of the 3-by-3 matrices are close to zero. This fact causes a considerable reduction in the magnitude of the round-off errors.

As a result of the numerical integration, the matrices $\overset{*}{M}_{ij}$, $\overset{*}{N}_{ij}$, $\overset{*}{S}_{ij}$, and $\overset{*}{T}_{ij}$ are found as a function of t_i for a fixed value of t_j and a known reference trajectory. Then, $\delta \underline{r}_i$ and $\delta \underline{v}_i$ are known in terms of the six constants that constitute $\delta \underline{x}_j$.

Matrices $\overset{*}{K}_{ji}$ and $\overset{*}{J}_{ji}$ may be evaluated by the use of Eqs. (F-34) and (F-33), respectively.

The eighteen-element matrices $\overset{*}{R}_j$, $\overset{*}{V}_j$, $\overset{*}{F}_i$, and $\overset{*}{L}_i$ cannot be evaluated, but they are not needed to solve the guidance problem. These matrices have been introduced because they illustrate the fact that each of the nine-element matrices may be regarded as the product of two matrices, one of which is a function of t_i only and the second of which is a function of only t_j. Moreover, it will be shown in Appendix K that the eighteen-element matrices are useful in deriving an analytic solution of the guidance problem when the reference trajectory is an ellipse.

F.7 Matrix Symmetry

In this section the following relations among the matrices are proved:

$$\overset{*}{T}_{ji} = \overset{*}{M}_{ij}{}^T \qquad \text{(F-75)}$$

$$\overset{*}{N}_{ji} = - \overset{*}{N}_{ij}{}^T \qquad \text{(F-76)}$$

$$\overset{*}{S}_{ji} = - \overset{*}{S}_{ij}{}^T \qquad \text{(F-77)}$$

$$\overset{*}{J}_{ij} = \overset{*}{J}_{ij}{}^T \qquad \text{(F-78)}$$

$$\overset{*}{K}_{ji} = - \overset{*}{K}_{ij}{}^T \qquad \text{(F-79)}$$

The superscript T signifies the transpose of a matrix. The proofs utilize the face that $\overset{*}{G}$ is known to be a symmetric matrix and $\overset{*}{W}$ is known to be skew-symmetric.

The first proof will be that of (F-78), which states that the $\overset{*}{J}_{ij}$ matrix is symmetric. This fact was first noted by Battin*. From Eqs. (F-33) and (F-34),

$$\overset{*}{M}_{ji}\,\overset{*}{J}_{ij}{}^{-1} = -\,\overset{*}{K}_{ij}{}^{-1} = -\,\overset{*}{N}_{ji} \qquad \text{(F-80)}$$

Equation (F-80) is differentiated with respect to t_i, with substitutions for the derivatives of $\overset{*}{M}_{ji}$ and $\overset{*}{N}_{ji}$ being made from (F-69) and (F-70).

$$\overset{*}{M}_{ji}\,\frac{\partial \overset{*}{J}_{ij}{}^{-1}}{\partial t_i} + (\overset{*}{M}_{ji}\,\overset{*}{W}_i - \overset{*}{N}_{ji}\,\overset{*}{G}_i)\overset{*}{J}_{ij}{}^{-1} = -\,\overset{*}{N}_{ji}\,\overset{*}{W}_i + \overset{*}{M}_{ji} \qquad \text{(F-81)}$$

(F-81) is pre-multiplied by $\overset{*}{M}_{ji}{}^{-1}$.

$$\frac{\partial \overset{*}{J}_{ij}{}^{-1}}{\partial t_i} + \overset{*}{J}_{ij}{}^{-1}\,\overset{*}{G}_i\,\overset{*}{J}_{ij}{}^{-1} + \overset{*}{W}_i\,\overset{*}{J}_{ij}{}^{-1} - \overset{*}{J}_{ij}{}^{-1}\,\overset{*}{W}_i = \overset{*}{I}_3 \qquad \text{(F-82)}$$

Since the left-hand side of (F-82) is equal to the identity matrix, which is symmetric, it must be equal to its own transpose. When (F-82) is equated to its transpose and $\overset{*}{G}_i{}^T$ and $\overset{*}{W}_i{}^T$ are replaced by $\overset{*}{G}_i$ and $-\,\overset{*}{W}_i$, respectively, the result is

$$\frac{\partial \overset{*}{J}_{ij}{}^{-1}}{\partial t_i} + \overset{*}{J}_{ij}{}^{-1}\,\overset{*}{G}_i\,\overset{*}{J}_{ij}{}^{-1} + \overset{*}{W}_i\,\overset{*}{J}_{ij}{}^{-1} - \overset{*}{J}_{ij}{}^{-1}\,\overset{*}{W}_i$$

$$= \frac{\partial \overset{*}{J}_{ij}{}^{-1^T}}{\partial t_i} + \overset{*}{J}_{ij}{}^{-1^T}\,\overset{*}{G}_i\,\overset{*}{J}_{ij}{}^{-1^T} + \overset{*}{W}_i\,\overset{*}{J}_{ij}{}^{-1^T} - \overset{*}{J}_{ij}{}^{-1^T}\,\overset{*}{W}_i \qquad \text{(F-83)}$$

---

* Page 697 of Reference (5)

It is apparent that this equation can be satisfied if $\overset{*}{J}_{ij}^{-1}$ is a symmetric matrix. It must now be proved that $\overset{*}{J}_{ij}^{-1}$ is necessarily symmetric.

Equation (F-82) consists of nine first-order differential equations, each of which has one constant of integration. If it can be shown that these constants are such that $\overset{*}{J}_{ij}^{-1}$ is symmetric at some particular time, then the matrix must be symmetric for all values of time.

When $t_i = t_j$,

$$\overset{*}{J}_{jj}^{-1} = - \overset{*}{M}_{jj}^{-1} \overset{*}{N}_{jj} = - \overset{*}{I}_3 \overset{*}{O}_3 = \overset{*}{O}_3 \tag{F-84}$$

The zero matrix is symmetric. Consequently, $\overset{*}{J}_{ij}^{-1}$ is symmetric when $t_i = t_j$ and hence for all values of time. The inverse of a non-singular symmetric matrix is itself symmetric. Therefore, the matrix $\overset{*}{J}_{ij}$ is symmetric for all combinations of t_i and t_j for which $\overset{*}{J}_{ij}^{-1}$ is non-singular, and Eq. (F-78) has been proved.

The second relation that will be derived is (F-76). Again the proof is a consequence of the symmetry of a matrix differential equation. Equation (F-53) is pre-multiplied by $-\overset{*}{N}_{ji}$. Subscripts i and j are interchanged in (F-70), and the resulting equation is post-multiplied by $\overset{*}{N}_{ij}$. These two equations are then added.

$$-\overset{*}{N}_{ji} \frac{\partial \overset{*}{N}_{ij}}{\partial t_i} + \frac{\partial \overset{*}{N}_{ji}}{\partial t_i} \overset{*}{N}_{ij} - 2 \overset{*}{N}_{ji} \overset{*}{W}_i \overset{*}{N}_{ij} = - \overset{*}{N}_{ji} \overset{*}{T}_{ij} - \overset{*}{M}_{ji} \overset{*}{N}_{ij} \tag{F-85}$$

From Eq. (F-40) it is seen that the right-hand side of (F-85) is equal to the zero matrix.

$$\frac{\partial \overset{*}{N}_{ji}}{\partial t_i} \overset{*}{N}_{ij} - \overset{*}{N}_{ji} \frac{\partial \overset{*}{N}_{ij}}{\partial t_i} = 2 \overset{*}{N}_{ji} \overset{*}{W}_i \overset{*}{N}_{ij} \tag{F-86}$$

Since W_i is skew-symmetric, the transpose of Eq. (F-86) is

$$- \frac{\partial \overset{*}{N}_{ij}{}^{T}}{\partial t_i} \overset{*}{N}_{ji}{}^{T} + \overset{*}{N}_{ij}{}^{T} \frac{\partial \overset{*}{N}_{ji}{}^{T}}{\partial t_i} = - 2 \overset{*}{N}_{ij}{}^{T} \overset{*}{W}_i \overset{*}{N}_{ji}{}^{T} \tag{F-87}$$

It is clear that both (F-86) and (F-87) can be satisfied if $\overset{*}{N}_{ji} = \pm \overset{*}{N}_{ij}{}^{T}$. The argument to be presented here is essentially the same as the one used in proving the symmetry of $\overset{*}{J}_{ij}$. Both (F-86) and (F-87) consist of first-order differential equations; therefore, if $\overset{*}{N}_{ji} = - \overset{*}{N}_{ij}{}^{T}$ at some particular time, the equality will be maintained for all values of time.

When $t_i = t_j$,

$$\overset{*}{M}_{jj} = \overset{*}{I}_3 \qquad \overset{*}{N}_{jj} = \overset{*}{O}_3 \qquad \overset{*}{T}_{jj} = \overset{*}{I}_3 \tag{F-88}$$

From (F-53),

$$\left. \frac{\partial \overset{*}{N}_{ij}}{\partial t_i} \right|_{t_i = t_j} = - \overset{*}{W}_j \overset{*}{N}_{jj} + \overset{*}{T}_{jj} = \overset{*}{I}_3 \tag{F-89}$$

From (F-70),

$$\left. \frac{\partial \overset{*}{N}_{ji}}{\partial t_i} \right|_{t_i = t_j} = \overset{*}{N}_{jj} \overset{*}{W}_j - \overset{*}{M}_{jj} = - \overset{*}{I}_3 \tag{F-90}$$

Equation (F-88) shows that when $t_i = t_j$, $\overset{*}{N}_{ji} = \pm \overset{*}{N}_{ij}{}^T$, since both are equal to $\overset{*}{O}_3$. Equations (F-89) and (F-90) are used to pick the proper sign; because the diagonal elements of $\overset{*}{N}_{ij}$, and hence of $\overset{*}{N}_{ij}{}^T$, are increasing with t_i, while the diagonal elements of $\overset{*}{N}_{ji}$ are decreasing with t_i, the negative sign is required.

$$\overset{*}{N}_{ji} = - \overset{*}{N}_{ij}{}^T \tag{F-91}$$

and (F-76) has been proved.

The proof of (F-79) follows directly from substituting (F-34) into (F-91) and then inverting and transposing both sides of the equation.

$$\overset{*}{N}_{ji} = - \overset{*}{N}_{ij}{}^T = \overset{*}{K}_{ij}{}^{-1} = - \overset{*}{K}_{ji}{}^{T^{-1}} \tag{F-92}$$

$$\overset{*}{K}_{ji} = - \overset{*}{K}_{ij}{}^T \tag{F-93}$$

(F-78) and (F-79) are used to establish (F-75). The left-hand part of Eq. (F-40) is solved for $\overset{*}{T}_{ji}$, and substitutions are made for $\overset{*}{M}_{ij}$, $\overset{*}{N}_{ji}$, and $\overset{*}{N}_{ij}$ from (F-33) and (F-34).

$$\begin{aligned}\overset{*}{T}_{ji} &= - \overset{*}{N}_{ij}{}^{-1} \overset{*}{M}_{ij} \overset{*}{N}_{ji} \\ &= - \overset{*}{K}_{ji} (- \overset{*}{K}_{ji}{}^{-1} \overset{*}{J}_{ji}) \overset{*}{K}_{ij}{}^{-1} \\ &= \overset{*}{J}_{ji} \overset{*}{K}_{ij}{}^{-1} = (- \overset{*}{K}_{ji}{}^{-1} \overset{*}{J}_{ji})^T \\ &= \overset{*}{M}_{ij}{}^T \end{aligned} \tag{F-94}$$

The proof that $\overset{*}{S}_{ji} = - \overset{*}{S}_{ij}{}^{T}$ involves the same sequence of steps as that used in deriving (F-76). Equation (F-52) is pre-multiplied by $- \overset{*}{S}_{ji} \overset{*}{G}_{i}{}^{-1}$. The subscripts in (F-71) are interchanged, and the equation is then post-multiplied by $\overset{*}{G}_{i}{}^{-1} \overset{*}{S}_{ij}$. The two resulting equations are added.

$$- \overset{*}{S}_{ji} \overset{*}{G}_{i}{}^{-1} \frac{\partial \overset{*}{S}_{ij}}{\partial t_i} - \overset{*}{S}_{ji} \overset{*}{G}_{i}{}^{-1} \overset{*}{W}_{i} \overset{*}{S}_{ij}$$

$$+ \frac{\partial \overset{*}{S}_{ji}}{\partial t_i} \overset{*}{G}_{i}{}^{-1} \overset{*}{S}_{ij} - \overset{*}{S}_{ji} \overset{*}{W}_{i} \overset{*}{G}_{i}{}^{-1} \overset{*}{S}_{ij}$$

$$= - \overset{*}{S}_{ji} \overset{*}{M}_{ij} - \overset{*}{T}_{ji} \overset{*}{S}_{ij} \qquad \text{(F-95)}$$

The right-hand side of (F-95) is the negative of the right-hand side of (F-40) and is therefore, equal to the zero matrix. (F-95) may then be simplified as follows:

$$\frac{\partial \overset{*}{S}_{ji}}{\partial t_i} \overset{*}{G}_{i}{}^{-1} \overset{*}{S}_{ij} - \overset{*}{S}_{ji} \overset{*}{G}_{i}{}^{-1} \frac{\partial \overset{*}{S}_{ij}}{\partial t_i}$$

$$= \overset{*}{S}_{ji} (\overset{*}{G}_{i}{}^{-1} \overset{*}{W}_{i} + \overset{*}{W}_{i} \overset{*}{G}_{i}{}^{-1}) \overset{*}{S}_{ij} \qquad \text{(F-96)}$$

The transpose of (F-96) is

$$- \frac{\partial \overset{*}{S}_{ij}{}^{T}}{\partial t_i} \overset{*}{G}_{i}{}^{-1} \overset{*}{S}_{ji}{}^{T} + \overset{*}{S}_{ij}{}^{T} \overset{*}{G}_{i}{}^{-1} \frac{\partial \overset{*}{S}_{ji}{}^{T}}{\partial t_i}$$

$$= - \overset{*}{S}_{ij}{}^{T} (\overset{*}{G}_{i}{}^{-1} \overset{*}{W}_{i} + \overset{*}{W}_{i} \overset{*}{G}_{i}{}^{-1}) \overset{*}{S}_{ji}{}^{T} \qquad \text{(F-97)}$$

Equations (F-96) and (F-97) can both be satisfied if

$$\overset{*}{S}_{ji} = \pm \overset{*}{S}_{ij}{}^{T}. \tag{F-98}$$

When $t_i = t_j$,

$$\overset{*}{S}_{jj} = \overset{*}{O}_3 = \pm \overset{*}{S}_{jj}{}^{T} \tag{F-99}$$

To determine the proper sign in (F-98) it is necessary to examine the derivatives of $\overset{*}{S}_{ij}$ and $\overset{*}{S}_{ji}$ with respect to t_i when $t_i = t_j$. From (F-52),

$$\left.\frac{\partial \overset{*}{S}_{ij}}{\partial t_i}\right|_{t_i = t_j} = - \overset{*}{W}_j \overset{*}{S}_{jj} + \overset{*}{G}_j \overset{*}{M}_{jj} = \overset{*}{G}_j \tag{F-100}$$

From (F-71)

$$\left.\frac{\partial \overset{*}{S}_{ji}}{\partial t_i}\right|_{t_i = t_j} = \overset{*}{S}_{jj} \overset{*}{W}_j - \overset{*}{T}_{jj} \overset{*}{G}_j = - \overset{*}{G}_j \tag{F-101}$$

Since $\overset{*}{G}$ is symmetric,

$$\left.\frac{\partial \overset{*}{S}_{ji}}{\partial t_i}\right|_{t_i = t_j} = - \left.\frac{\partial \overset{*}{S}_{ij}{}^{T}}{\partial t_i}\right|_{t_i = t_j} \tag{F-102}$$

It follows from (F-96), (F-97), (F-99), and (F-102) that

$$\overset{*}{S}_{ji} = - \overset{*}{S}_{ij}{}^{T} \tag{F-103}$$

All five of the relations stated at the beginning of this section have now been proved.

By the use of the first three relations, the 6-by-6 matrix $\overset{*}{C}_{ij}$ may be inverted by inspection.

$$\overset{*}{C}_{ij}{}^{-1} = \overset{*}{C}_{ji} = \begin{Bmatrix} \overset{*}{M}_{ji} & \overset{*}{N}_{ji} \\ \overset{*}{S}_{ji} & \overset{*}{T}_{ji} \end{Bmatrix} = \begin{Bmatrix} \overset{*}{T}_{ij}{}^{T} & -\overset{*}{N}_{ij}{}^{T} \\ -\overset{*}{S}_{ij}{}^{T} & \overset{*}{M}_{ij}{}^{T} \end{Bmatrix} \tag{F-104}$$

F. 8 Method of Adjoints

Since the completion of the work reported in the last section, the author has been apprised of two additional methods of proving the inverse relationship of Eq. (F-104). These are included here to round out the discussion of matrix formulations. The first method employs adjoint functions, and the second involves the properties of symplectic matrices.

The adjoint method is suggested in the work of McLean, Schmidt, and McGee(13). The technique requires that Eq. (E-13), which consists of three second-order equations, be re-cast as a set of six first-order equations. This is accomplished as follows:

$$\delta \dot{\underline{x}} = \begin{Bmatrix} \delta \dot{\underline{r}} \\ \delta \dot{\underline{v}} \end{Bmatrix} = \begin{Bmatrix} \overset{*}{O}_3 & \overset{*}{I}_3 \\ \overset{*}{G} & \overset{*}{O}_3 \end{Bmatrix} \begin{Bmatrix} \delta \underline{r} \\ \delta \underline{v} \end{Bmatrix} = \overset{*}{Z} \, \delta \underline{x} \tag{F-105}$$

where

$$\overset{*}{Z} = \begin{Bmatrix} \overset{*}{O}_3 & \overset{*}{I}_3 \\ \overset{*}{G} & \overset{*}{O}_3 \end{Bmatrix} \tag{F-106}$$

The vector $\underline{\lambda}$, which is adjoint to $\delta\underline{x}$, is defined by the matrix equation

$$\dot{\underline{\lambda}} = - \overset{*}{Z}^T \underline{\lambda} \tag{F-107}$$

The six-component vector $\underline{\lambda}$ may be partitioned into two three-component vectors $\underline{\mu}$ and $\underline{\nu}$.

$$\dot{\underline{\lambda}} = \begin{Bmatrix} \dot{\underline{\mu}} \\ \dot{\underline{\nu}} \end{Bmatrix} = \begin{Bmatrix} \overset{*}{O}_3 & -\overset{*}{G}^T \\ -\overset{*}{I}_3 & \overset{*}{O}_3 \end{Bmatrix} \begin{Bmatrix} \underline{\mu} \\ \underline{\nu} \end{Bmatrix} \tag{F-108}$$

Since $\overset{*}{G}$ is a symmetric matrix,

$$\dot{\underline{\mu}} = - \overset{*}{G} \underline{\nu} \tag{F-109}$$

Also,

$$\dot{\underline{\nu}} = - \underline{\mu} \tag{F-110}$$

(F-109) and (F-110) can be combined into a single second-order vector equation.

$$\ddot{\underline{\nu}} = \overset{*}{G} \underline{\nu} \tag{F-111}$$

This equation has the same form as (E-13).

As a consequence of (F-110), $\underline{\lambda}$ may be written as

$$\underline{\lambda} = \begin{Bmatrix} \underline{\mu} \\ \underline{\nu} \end{Bmatrix} = \begin{Bmatrix} -\dot{\underline{\nu}} \\ \underline{\nu} \end{Bmatrix} \tag{F-112}$$

To show the relation between $\delta \underline{x}$ and $\underline{\lambda}$, pre-multiply (F-105) by $\underline{\lambda}^T$, post-multiply the transpose of (F-107) by $\delta \underline{x}$, and add.

$$\underline{\lambda}^T \delta \dot{\underline{x}} + \dot{\underline{\lambda}}^T \delta \underline{x} = \underline{\lambda}^T \overset{*}{Z} \delta \underline{x} - \underline{\lambda}^T \overset{*}{Z} \delta \underline{x} = 0 \tag{F-113}$$

$$\frac{d}{dt} (\underline{\lambda}^T \delta \underline{x}) = 0 \tag{F-114}$$

$$\underline{\lambda}^T \delta \underline{x} = \text{constant} \tag{F-115}$$

Like $\delta \underline{x}$, $\underline{\lambda}$ must be a time-varying vector. By analogy with (F-35) $\underline{\lambda}_j$; the value of $\underline{\lambda}$ at t_j, may be related to $\underline{\lambda}_i$.

$$\underline{\lambda}_j = \overset{*}{D}_{ji} \underline{\lambda}_i \tag{F-116}$$

From (F-115),

$$\underline{\lambda}_j^T \delta \underline{x}_j = \underline{\lambda}_i^T \delta \underline{x}_i \tag{F-117}$$

(F-35) and (F-116) are substituted into (F-117).

$$\underline{\lambda}_i^T \overset{*}{D}_{ji}{}^T \overset{*}{C}_{ji} \delta \underline{x}_i = \underline{\lambda}_i^T \delta \underline{x}_i \tag{F-118}$$

Since $\delta \underline{x}_i$ is arbitrary and $\underline{\lambda}_i$ is assumed not to be a zero vector,

$$\overset{*}{D}_{ji}{}^{T} \overset{*}{C}_{ji} = \overset{*}{I}_6 \tag{F-119}$$

$$\overset{*}{C}_{ji}{}^{-1} = \overset{*}{D}_{ji}{}^{T} \tag{F-120}$$

Equation (F-120) relates the 6-by-6 solution matrix of (F-35) to the 6-by-6 solution matrix of (F-116).

A new six-component vector $\underline{\lambda}'$ is defined as follows:

$$\underline{\lambda}' = \begin{Bmatrix} \underline{\nu} \\ \dot{\underline{\nu}} \end{Bmatrix} \tag{F-121}$$

$\underline{\lambda}'$ is related to $\underline{\lambda}$ by the skew-symmetric matrix $\overset{*}{P}$.

$$\underline{\lambda}' = \overset{*}{P} \, \underline{\lambda} \tag{F-122}$$

where

$$\overset{*}{P} = \begin{Bmatrix} \overset{*}{O}_3 & \overset{*}{I}_3 \\ -\overset{*}{I}_3 & \overset{*}{O}_3 \end{Bmatrix} \tag{F-123}$$

$\overset{*}{P}$ has some interesting properties.

$$\overset{*}{P}{}^2 = -\overset{*}{I}_3 \tag{F-124}$$

$$\overset{*}{P}{}^{-1} = \overset{*}{P}{}^{T} = -\overset{*}{P} \tag{F-125}$$

Equation (F-107) can now be written in terms of $\underline{\lambda}'$.

$$\dot{\underline{\lambda}} = \overset{*}{P}^{-1} \dot{\underline{\lambda}}' = - \overset{*}{Z}^{T} \overset{*}{P}^{-1} \underline{\lambda}' \tag{F-126}$$

This equation is pre-multiplied by $\overset{*}{P}$.

$$\dot{\underline{\lambda}}' = \overset{*}{P} \overset{*}{Z}^{T} \overset{*}{P} \underline{\lambda}' \tag{F-127}$$

When the matrix multiplication in (F-127) is carried out, it is found that

$$\overset{*}{P} \overset{*}{Z}^{T} \overset{*}{P} = \overset{*}{Z} \tag{F-128}$$

$$\therefore \dot{\underline{\lambda}}' = Z \underline{\lambda}' \tag{F-129}$$

The form of (F-129) is identical with that of (F-105). Therefore, the solution for $\underline{\lambda}'$ must be the same as that for $\delta\underline{x}$ except for a difference in the six arbitrary constants. The constants for $\underline{\lambda}'$ are the components of $\underline{\lambda}_i'$. Then, by analogy with (F-35),

$$\underline{\lambda}_j' = \overset{*}{C}_{ji} \underline{\lambda}_i' \tag{F-130}$$

From (F-116), (F-122), and (F-130),

$$\underline{\lambda}_j' = \overset{*}{P} \underline{\lambda}_j = \overset{*}{C}_{ji} \overset{*}{P} \underline{\lambda}_i = \overset{*}{P} \overset{*}{D}_{ji} \underline{\lambda}_i \tag{F-131}$$

For an arbitrary $\underline{\lambda}_i$,

$$\overset{*}{C}_{ji} \overset{*}{P} = \overset{*}{P} \overset{*}{D}_{ji} \tag{F-132}$$

$$\overset{*}{D}_{ji} = \overset{*}{P}^{-1} \overset{*}{C}_{ji} \overset{*}{P} = - \overset{*}{P} \overset{*}{C}_{ji} \overset{*}{P} \tag{F-133}$$

From (F-120), the inverse of $\overset{*}{C}_{ji}$ is

$$\overset{*}{C}_{ji}{}^{-1} = \overset{*}{D}_{ji}{}^{T} = - \overset{*}{P}\,\overset{*}{C}_{ji}{}^{T}\,\overset{*}{P} \tag{F-134}$$

The matrix multiplication is carried out by use of the definitions given in (F-36) and (F-123).

$$\overset{*}{C}_{ji}{}^{-1} = \left\{ \begin{array}{cc} \overset{*}{T}_{ji}{}^{T} & -\overset{*}{N}_{ji}{}^{T} \\ -\overset{*}{S}_{ji}{}^{T} & \overset{*}{M}_{ji}{}^{T} \end{array} \right\} \tag{F-135}$$

This equation is the equivalent of (F-104).

F.9 Symplectic Matrices

The author is indebted to Dr. James E. Potter, of the staff of the M.I.T. Instrumentation Laboratory, who first pointed out to him that the transition matrix is symplectic and that this fact can be exploited in studying the properties of the transition matrix. A mathematically rigorous discussion of symplectic groups is presented in Chapter VI of Weyl.[42]

A symplectic matrix can be defined by analogy with an orthogonal matrix. The matrix $\overset{*}{A}$ is orthogonal if

$$\overset{*}{A}{}^{T}\,\overset{*}{I}\,\overset{*}{A} = \overset{*}{I} \tag{F-136}$$

$\overset{*}{I}$ being the familiar identity matrix. The matrix $\overset{*}{Y}$ is symplectic if

$$\overset{*}{A}{}^{T}\,\overset{*}{P}\,\overset{*}{A} = \overset{*}{P} \tag{F-137}$$

where $\overset{*}{P}$ is given by (F-123).

It will now be shown that $\overset{*}{C}_{ji}$ is symplectic. The time derivative of the scalar quantity $\delta\underline{x}^T \overset{*}{P} \delta\underline{x}$ is

$$\frac{d}{dt}(\delta\underline{x}^T \overset{*}{P} \delta\underline{x}) = \delta\dot{\underline{x}}^T P \delta\underline{x} + \delta\underline{x}^T P \delta\dot{\underline{x}}$$

$$= \delta\underline{x}^T \overset{*}{Z}^T \overset{*}{P} \delta\underline{x} + \delta\underline{x}^T \overset{*}{P} \overset{*}{Z} \delta\underline{x}$$

$$= \delta\underline{x}^T (\overset{*}{Z}^T \overset{*}{P} + \overset{*}{P} \overset{*}{Z}) \delta\underline{x} \tag{F-138}$$

From the definitions of $\overset{*}{Z}$ and $\overset{*}{P}$ and the fact that $\overset{*}{G}$ is symmetric, it can be shown that $(\overset{*}{Z}^T \overset{*}{P} + \overset{*}{P} \overset{*}{Z})$ is equal to the 6-by-6 zero matrix. Then,

$$\frac{d}{dt}(\delta\underline{x}^T \overset{*}{P} \delta\underline{x}) = 0 \tag{F-139}$$

$$\delta\underline{x}_j^T \overset{*}{P} \delta\underline{x}_j = \delta\underline{x}_i^T \overset{*}{P} \delta\underline{x}_i = \text{constant} \tag{F-140}$$

(F-35) is substituted into (F-140).

$$\delta\underline{x}_i^T \overset{*}{C}_{ji}{}^T \overset{*}{P} \overset{*}{C}_{ji} \delta\underline{x}_i = \delta\underline{x}_i^T \overset{*}{P} \delta\underline{x}_i \tag{F-141}$$

Since $\delta\underline{x}_i$ is arbitrary,

$$\overset{*}{C}_{ji}{}^T \overset{*}{P} \overset{*}{C}_{ji} = \overset{*}{P} \tag{F-142}$$

and hence $\overset{*}{C}_{ji}$ is a symplectic matrix.

To find $\overset{*}{C}_{ji}{}^{-1}$, (F-142) is pre-multiplied by $-\overset{*}{P}$ and post-multiplied by $\overset{*}{C}_{ji}{}^{-1}$.

$$\overset{*}{C}_{ji}{}^{-1} = -\overset{*}{P} \overset{*}{C}_{ji}{}^T \overset{*}{P} \tag{F-143}$$

Equation (F-143) is the same as (F-134). Thus, the triple matrix product of (F-143) leads to the expression for $\overset{*}{C}_{ji}^{-1}$ given by (F-135).

Equation (F-142) may be used to evaluate the determinant of $\overset{*}{C}_{ji}$. Since the determinant of a matrix is equal to the determinant of its transpose,

$$(\det \overset{*}{C}_{ji})(\det \overset{*}{P})(\det \overset{*}{C}_{ji}) = \det \overset{*}{P} \qquad \text{(F-144)}$$

From the definition of $\overset{*}{P}$ in (F-123),

$$\det \overset{*}{P} = +1 \qquad \text{(F-145)}$$

Then,

$$(\det \overset{*}{C}_{ji})^2 = 1 \qquad \text{(F-146)}$$

$$\det \overset{*}{C}_{ji} = \pm 1 \qquad \text{(F-147)}$$

Since $\overset{*}{C}_{ii} = \overset{*}{I}_6$ and the elements of $\overset{*}{C}_{ji}$ are continuous functions of time, the plus sign is required in (F-147).

$$\det \overset{*}{C}_{ji} = +1 \qquad \text{(F-148)}$$

This equation is useful in checking the numerical evaluation of the elements of $\overset{*}{C}_{ji}$.

APPENDIX G

INTEGRATION OF THE VARIANT EQUATIONS OF MOTION FOR ELLIPTICAL REFERENCE TRAJECTORIES

G.1 Summary

The variant equations of motion are developed for the two-body problem. The system consists of three second-order linear differential equations with variable coefficients. By choosing as a set of coordinate axes one of the reference trajectory sets of Appendix A, the sixth-order system is sub-divided into two uncoupled systems, one of fourth order and the other of second order. The two uncoupled systems are integrated directly to yield position variation relative to the reference trajectory.

G.2 Variant Equations for Two-Body Motion

The variant equations for many-body motion are developed in Appendix E. The matrix equations in the three reference trajectory coordinate systems are (E-11), (E-18), and (E-19). For two-body motion the equations are considerably simplified by the removal of all effects of disturbing forces.

Just as the r s z coordinate system was used to integrate the general equations of two-body motion in Appendix B, so it has been found that the same coordinate system is most effective in integrating the variant equations of two-body motion. When the disturbing forces are neglected in Eq. (E-18), z is equal to zero, and ρ may be replaced by r. Then the variant equations become

$$\begin{pmatrix} \ddot{\delta r} - \dot{f}^2 \delta r - 2\dot{f}\, \dot{\delta s} - \ddot{f}\, \delta s \\ 2\dot{f}\, \dot{\delta r} + \ddot{f}\, \delta r + \ddot{\delta s} - \dot{f}^2\, \delta s \\ \ddot{\delta z} \end{pmatrix} = \frac{\mu}{r^3} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \delta r \\ \delta s \\ \delta z \end{pmatrix} \tag{G-1}$$

This equation can be simplified by expressing δs and its derivatives in terms of δf and its derivatives.

$$\delta s = r\, \delta f \tag{G-2}$$

$$\dot{\delta s} = r\, \dot{\delta f} + \dot{r}\, \delta f \tag{G-3}$$

$$\ddot{\delta s} = r\, \ddot{\delta f} + 2\, \dot{r}\, \dot{\delta f} + \ddot{r}\, \delta f \tag{G-4}$$

These three equations are substituted into the left-hand side of (G-1).

$$\begin{pmatrix} \ddot{\delta r} - \dot{f}^2 \delta r - 2\, r\, \dot{f}\, \dot{\delta f} - (2\, \dot{r}\, \dot{f} + r\, \ddot{f})\, \delta f \\ 2\, \dot{f}\, \dot{\delta r} + \ddot{f}\, \delta r + r\, \ddot{\delta f} + 2\, \dot{r}\, \dot{\delta f} + (\ddot{r} - r\, \dot{f}^2)\, \delta f \\ \ddot{\delta z} \end{pmatrix}$$

$$= \frac{\mu}{r^3} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \delta r \\ \delta s \\ \delta z \end{pmatrix} \tag{G-5}$$

The equations of (B-32) are substituted into (G-5).

$$\begin{pmatrix} \delta\ddot{r} - \dot{f}^2\,\delta r - 2\,r\,\dot{f}\,\delta\dot{f} \\ 2\,\dot{f}\,\delta\dot{r} + \ddot{f}\,\delta r + r\,\delta\ddot{f} + 2\,\dot{r}\,\delta\dot{f} \\ \delta\ddot{z} \end{pmatrix} = \frac{\mu}{r^3}\begin{pmatrix} 2\,\delta r \\ 0 \\ -\,\delta z \end{pmatrix} \tag{G-6}$$

It is immediately apparent from (G-6) that the variant motion in the reference trajectory plane and the variant motion perpendicular to that plane are completely independent of each other. Therefore, the two types of motion will be studied separately.

G.3 Three Solutions for Motion in Reference Trajectory Plane

The motion in the reference trajectory plane will be investigated first. This motion involves the first two equations of (G-6). The two equations are coupled equations in the variables δr and δf. They may be re-written as follows:

$$(D^2 - \dot{f}^2 - \frac{2\mu}{r^3})\,\delta r - 2\,r\,\dot{f}\,D\,\delta f = 0 \tag{G-7}$$

$$(2\,\dot{f}\,D + \ddot{f})\,\delta r + (r\,D + 2\,\dot{r})\,D\,\delta f = 0 \tag{G-8}$$

where the operator D is equal to $\frac{d}{dt}$.

These two equations constitute a fourth-order system in the variables δr and δf. Since δf itself does not appear in either equation, the system may be regarded as third-order in the variables δr and $D\,\delta f$.

The fact that δf does not appear in either equation indicates that δr is dependent on only the derivatives of δf, not on δf itself. One solution of the coupled equations is then

$$\delta r = 0 \qquad \delta f = k_1 \tag{G-9}$$

where k_1 is an arbitrary constant.

The solution of the third-order system of (G-7) and (G-8) is expedited if the independent variable is changed from t to f. The symbol F is used to represent $\frac{d}{df}$. The following substitutions may be made:

$$D = \dot{f}\, F \tag{G-10}$$

$$D^2 = \dot{f}\, F\, (\dot{f}\, F) \tag{G-11}$$

From (B-58),

$$F\,\dot{f} = \frac{-\,2\,n\,e\,\sin f\,(1 + e\cos f)}{(1 - e^2)^{3/2}}$$

$$= -\,\frac{2\,e\,\sin f}{1 + e\cos f}\,\dot{f} \tag{G-12}$$

(G-12) is substituted into (G-11).

$$D^2 = \dot{f}^2 \left(F^2 - \frac{2\, e \sin f}{1 + e \cos f}\, F\right) \qquad \text{(G-13)}$$

From (B-39), (B-60), and (B-61),

$$-\frac{2\mu}{r^3} = -\frac{2\,\mu\, r\, \dot{f}^2}{h^2} = -\frac{2\,\dot{f}^2}{1 + e \cos f} \qquad \text{(G-14)}$$

The coefficient of δf in (G-7) is

$$-2\, r\, \dot{f}\, D = -\frac{2\, a\, (1 - e^2)\, \dot{f}^2}{1 + e \cos f}\, F \qquad \text{(G-15)}$$

Equations (G-13), (G-14), and (G-15) are incorporated into (G-7), and the resulting equation is multiplied by

$$\frac{(1 + e \cos f)}{\dot{f}^2}$$

$$\left[(1 + e \cos f)\, F^2 - (2\, e \sin f)\, F - (3 + e \cos f)\right] \delta r$$

$$-\, 2\, a\, (1 - e^2)\, F\, \delta f = 0 \qquad \text{(G-16)}$$

The coefficient of δr in (G-8) is

$$(2\dot{f}\, D + \ddot{f}) = 2\,\dot{f}^2 \left(F - \frac{e \sin f}{1 + e \cos f}\right) \qquad \text{(G-17)}$$

With the aid of (B-65) and (B-66), the coefficient of δf in (G-8) may be written as follows:

$$r\, D^2 + 2\,\dot{r}\, D = r\,\dot{f}^2 \left(F^2 - \frac{2 e \sin f}{1 + e \cos f}\, F\right) + 2\,\dot{r}\,\dot{f}\, F$$

$$= r\,\dot{f}^2\, F^2 + 2\dot{f}\left(\dot{r} - \frac{e \sin f}{1 + e \cos f}\, r\,\dot{f}\right) F$$

$$= r\,\dot{f}^2\, F^2 \qquad \text{(G-18)}$$

(G-17) and (G-18) are substituted into (G-8), and this equation, like (G-2) is multiplied by

$$\frac{(1 + e \cos f)}{\dot{f}^2}.$$

$$2\left[(1 + e \cos f)\, F - e \sin f\right] \delta r + a\,(1 - e^2)\, F^2\, \delta f = 0 \qquad \text{(G-19)}$$

The variable δf may be eliminated from the coupled equations (G-16) and (G-19) by pre-multiplying the former by the operator F, multiplying the latter by 2, and then adding.

$$[(1 + e \cos f)\, F^3 - (3\, e \sin f)\, F^2$$

$$+ (1 + e \cos f)\, F - (3\, e \sin f)]\, \delta r = 0 \qquad \text{(G-20)}$$

The terms of (G-20) may be re-grouped as follows:

$$[(1 + e \cos f)\, F - (3\, e \sin f)]\, (F^2 + 1)\, \delta r = 0 \qquad \text{(G-21)}$$

Two solutions of (G-16) are obtained from

$$(F^2 + 1)\, \delta r = 0 \qquad \text{(G-22)}$$

These solutions are obviously

$$\delta r = k_2 \cos f \qquad \text{(G-23)}$$

$$\delta r = k_3 \sin f \qquad \text{(G-24)}$$

The solution of (G-23) is substituted into (G-16) in order to solve for $F\, \delta f$.

$$F\, \delta f = - \frac{k_2}{a\,(1 - e^2)}\, [2 \cos f + e\, (\cos^2 f - \sin^2 f)] \qquad \text{(G-25)}$$

Then δf is obtained by integration.

$$\delta f = - \frac{k_2}{a\,(1 - e^2)}\, (2 + e \cos f) \sin f \qquad \text{(G-26)}$$

The solution of (G-24) is handled in similar fashion.

$$F\ \delta f = -\frac{2\,k_3}{a\,(1 - e^2)}\,(1 + e \cos f) \sin f \tag{G-27}$$

$$\delta f = \frac{k_3}{a\,(1 - e^2)}\,(2 + e \cos f) \cos f \tag{G-28}$$

G.4 Fourth Solution for Motion in Reference Trajectory Plane

The first three solutions of Eqs. (G-7) and (G-8) were obtained relatively easily. The fourth solution requires considerably more mathematical manipulation.

One technique for obtaining the fourth solution is to substitute the two known solutions, (G-23) and (G-24), successively into (G-20) and by so doing to reduce (G-20) from a third-order equation to a first-order equation, which can be solved directly by the use of an integrating factor. A method which might be considered mathematically more elegant is the method of variation of parameters. Both methods are described in detail in the first chapter of Hildebrand[41]. The second method is used in the following analysis.

In Eq. (G-21), let

$$x = (F^2 + 1)\,\delta r \tag{G-29}$$

Then (G-21) may be written as follows:

$$\frac{dx}{df} - \frac{3\,e \sin f}{1 + e \cos f}\,x = 0 \tag{G-30}$$

The variables x and f are now separable.

$$\frac{dx}{x} - \frac{3\ e\ \sin f}{1 + e \cos f}\ df = 0 \tag{G-31}$$

This equation may be integrated directly. The result of the integration is

$$\log x + 3 \log (1 + e \cos f) = \log C \tag{G-32}$$

where C is an arbitrary constant. Then,

$$x = (F^2 + 1)\ \delta r = \frac{C}{(1 + e \cos f)^3} \tag{G-33}$$

Since the two homogeneous solutions of (G-33) are known to be cos f and sin f, the method of variation of parameters may be used to get the particular solution of (G-33). In this method, the solution is assumed to be of the form.

$$\delta r = u \cos f + v \sin f \tag{G-34}$$

where u and v are functions of f. The variables u and v must satisfy the following two criteria:

$$\frac{du}{df} \cos f + \frac{dv}{df} \sin f = 0 \tag{G-35}$$

$$\frac{du}{df}\ \frac{d\,(\cos f)}{df} + \frac{dv}{df}\ \frac{d\,(\sin f)}{df}$$

$$= -\frac{du}{df} \sin f + \frac{dv}{df} \cos f = \frac{C}{(1 + e \cos f)^3} \tag{G-36}$$

The two simultaneous equations (G-35) and (G-36) are solved for $\frac{du}{df}$ and $\frac{dv}{df}$.

$$\frac{du}{df} = - \frac{C \sin f}{(1 + e \cos f)^3} \tag{G-37}$$

$$\frac{dv}{df} = \frac{C \cos f}{(1 + e \cos f)^3} \tag{G-38}$$

Equation (G-37) may be integrated directly.

$$u = - C \int \frac{\sin f \; df}{(1 + e \cos f)^3} = \frac{C}{e} \frac{d\,(1 + e \cos f)}{(1 + e \cos f)^3}$$

$$= - \frac{C}{2\,e\,(1 + e \cos f)^2} \tag{G-39}$$

The integration of (G-38) is less obvious. It is desirable to remove the polynomial in the denominator by making a change of variable from the true anomaly f to the eccentric anomaly E. Equations (B-52), (B-54), and (B-64) are used in making the change.

$$v = C\int \frac{\cos f \; df}{(1 + e\cos f)^3}$$

$$= C\int \frac{\cos E - e}{1 - e\cos E} \cdot \frac{(1 - e\cos E)^3}{(1 - e^2)^3} \cdot \frac{(1 - e^2)^{1/2}}{1 - e\cos E} \; dE$$

$$= \frac{C}{(1 - e^2)^{5/2}} \int (1 - e\cos E)(\cos E - e) \; dE$$

$$= \frac{C}{(1 - e^2)^{5/2}} \int \left[-e + (1 + e^2)\cos E - e\cos^2 E \right] dE \qquad \text{(G-40)}$$

The individual terms of (G-40) can now be integrated.

$$v = \frac{C}{(1 - e^2)^{5/2}} \left[-eE + (1 + e^2)\sin E - \frac{e}{2}(E + \sin E\cos E) \right]$$

$$= \frac{C}{2(1 - e^2)^{5/2}} \left\{ -3eE + \left[2(1 + e^2) - e\cos E \right] \sin E \right\} \qquad \text{(G-41)}$$

No constants of integration have been added in (G-39) and (G-41) because such constants, which would simply be multiplied by cos f and sin f respectively, may be incorporated into the constants k_2 and k_3 of Eqs. (G-23) and (G-24).

In (G-41), the eccentric anomaly E may be written in terms of the mean anomaly M and sin E by the use of Kepler's equation (B-55). Then the terms in E may be converted back to functions of f by using (B-53) and (B-54).

$$v = \frac{C}{2\,(1-e^2)^{5/2}} \left\{ -3e\left[M + \frac{e\,(1-e^2)^{1/2}\sin f}{1+e\cos f}\right] \right.$$

$$\left. + \left[2\,(1+e^2) - \frac{e\,(\cos f + e)}{1+e\cos f}\right] \frac{(1-e^2)^{1/2}\sin f}{1+e\cos f} \right\}$$

$$= \frac{C}{2\,(1-e^2)} \left[-\frac{3\,e\,M}{(1-e^2)^{3/2}} + \frac{(2+e\cos f)\sin f}{(1+e\cos f)^2} \right] \qquad \text{(G-42)}$$

Equations (G-39) and (G-42) are substituted into (G-34) to yield the fourth solution for δr.

$$\delta r = C\left[-\frac{\cos f}{2e\,(1+e\cos f)^2} - \frac{3\,e\,M\sin f}{2\,(1-e^2)^{5/2}} \right.$$

$$\left. + \frac{(2+e\cos f)\sin^2 f}{2(1-e^2)(1+e\cos f)^2} \right]$$

$$= \frac{C}{1-e^2}\left[-\frac{3\,e\,M\sin f}{2\,(1-e^2)^{3/2}} + \frac{1}{1+e\cos f} - \frac{\cos f}{2e} \right] \qquad \text{(G-34)}$$

The term

$$-\frac{C}{2e\,(1-e^2)}\cos f$$

in Eq. (G-43) may be incorporated into the constant k_2 of Eq. (G-23), so that the fourth solution becomes, finally,

$$\delta r = k_4 \left[- \frac{3\, e\, M \sin f}{2\,(1-e^2)^{3/2}} + \frac{1}{1 + e \cos f} \right] \tag{G-44}$$

where

$$k_4 = \frac{C}{1 - e^2} \tag{G-45}$$

To determine the fourth solution for δf from (G-16), the first and second derivatives of (G-44) with respect to f must be found. The derivative of M with respect to f is obtained from (B-56) and (B-58).

$$F\,(M) = \frac{\dot{M}}{\dot{f}} = \frac{(1-e^2)^{3/2}}{(1 + e \cos f)^2} \tag{G-46}$$

The derivatives of (G-44) are

$$F(\delta r) = k_4 \left\{ - \frac{3e}{2\,(1-e^2)^{3/2}} \left[M \cos f + \frac{(1-e^2)^{3/2}}{(1 + e \cos f)^2} \sin f \right] + \frac{e \sin f}{(1 + e \cos f)^2} \right\}$$

$$= -\, k_4 \left[\frac{3\, e\, M \cos f}{2\,(1-e^2)^{3/2}} + \frac{e \sin f}{2\,(1 + e \cos f)^2} \right] \tag{G-47}$$

$$F^2(\delta r) = -k_4 \left\{ \frac{3e}{2(1-e^2)^{3/2}} \left[-M \sin f + \frac{(1-e^2)^{3/2}\cos f}{(1+e\cos f)^2} \right] \right.$$

$$\left. + \frac{e \cos f}{2(1+e\cos f)^2} + \frac{e^2 \sin^2 f}{(1+e\cos f)^3} \right\}$$

$$= k_4 \left[\frac{3\,e\,M \sin f}{2(1-e^2)^{3/2}} - \frac{1}{1+e\cos f} + \frac{1-e^2}{(1+e\cos f)^3} \right] \tag{G-48}$$

When (G-47) and (G-48) are substituted into (G-16), the resulting expression for $F(\delta f)$ is

$$F(\delta f) = \frac{3\,k_4}{2a(1-e^2)} \left[\frac{2e\,M(1+e\cos f)\sin f}{(1-e^2)^{3/2}} - 1 \right] \tag{G-49}$$

Integration by parts is used to solve for δf from (G-49). Note that

$$dM = (1 - e\cos E)\,dE = \frac{(1-e^2)^{3/2}\,df}{(1+e\cos f)^2} \tag{G-50}$$

Therefore,

$$e\,M(1+e\cos f)\sin f\,df = -M(1+e\cos f)\,d(1+e\cos f)$$

$$= -d\left[\tfrac{1}{2} M (1+e\cos f)^2\right] + \tfrac{1}{2}(1+e\cos f)^2\,dM$$

$$= -d\left[\tfrac{1}{2} M (1+e\cos f)^2\right] + \tfrac{1}{2}(1-e^2)^{3/2}\,df \tag{G-51}$$

The integral of (G-49) is simply

$$\delta f = - \frac{3 k_4}{2 a (1 - e^2)^{5/2}} M (1 + e \cos f)^2 \qquad \text{(G-52)}$$

No constant of integration is needed in deriving δf from $\mathbf{F}(\delta f)$ because of the presence of the constant k_1, which is the first solution of δf.

G. 5 Solutions for Motion Normal to Reference Trajectory Plane

The differential equation for the motion normal to the reference trajectory plane is the third equation of (G-6).

$$(D^2 + \frac{\mu}{r^3}) \delta z = 0 \qquad \text{(G-53)}$$

To solve for δz, the independent variable is changed from t to the eccentric anomaly E. The symbol J is used for the operator $\frac{d}{dE}$. The operator D^2 in (G-53) can be expressed in terms of J and J^2.

$$D = \dot{E} J \qquad \text{(G-54)}$$

$$D^2 = \dot{E} J (\dot{E} J) = \dot{E} [\dot{E} J^2 + \frac{d\dot{E}}{dE} J] \qquad \text{(G-55)}$$

From (B-57),

$$\frac{d\dot{E}}{dE} = - \frac{n e \sin E}{(1 - e \cos E)^2} = - \frac{e \sin E}{1 - e \cos E} \dot{E} \qquad \text{(G-56)}$$

Then,

$$D^2 = \frac{\dot{E}^2}{(1 - e \cos E)} \left[(1 - e \cos E)\, J^2 - (e \sin E)\, J \right] \qquad \text{(G-57)}$$

From (B-49), (B-57), and (B-62),

$$\frac{\mu}{r^3} = \frac{n^2 a^3}{a^3 (1 - e \cos E)^3} = \frac{\dot{E}^2}{(1 - e \cos E)} \qquad \text{(G-58)}$$

(G-57) and (G-58) are substituted into (G-53), and the equation is multiplied by

$$\frac{1 - e \cos E}{\dot{E}^2} .$$

$$\left[(1 - e \cos E)\, J^2 - (e \sin E)\, J + 1 \right] \delta z = 0 \qquad \text{(G-59)}$$

The terms in (G-59) may be re-arranged as follows:

$$\left[(J^2 + 1) - e\, (\cos E\, J + \sin E)\, J \right] \delta z = 0 \qquad \text{(G-60)}$$

From the appearance of (G-60), two possible trial solutions for δz are immediately suggested, namely, sin E and cos E. It is found that sin E is indeed a solution. However, when $\delta z = \cos E$ is tried, the result is

$$\left[(J^2 + 1) - e\, (\cos E\, J + \sin E)\, J \right] (\cos E) = e \qquad \text{(G-61)}$$

Since e is a constant and since the coefficient of the undifferentiated term in (G-59) is unity, the second solution is (cos E - e).

The two solutions may be expressed in terms of the true anomaly f by making use of (B-52), (B-53), and (B-54).

$$\delta z = \frac{k_5}{(1 - e^2)^{1/2}} \sin E = k_5 \frac{\sin f}{1 + e \cos f} \tag{G-62}$$

$$\delta z = \frac{k_6}{(1 - e^2)} (\cos E - e) = k_6 \frac{\cos f}{1 + e \cos f} \tag{G-63}$$

G. 6 Complete Solution for Position Variation

The results of this appendix may be summarized by tabulating the complete solution for the position variation vector $\delta \underline{r}$ in the r s z coordinate system. The component δs, in the transverse direction, is related to δf by the equation

$$\delta s = r \ \delta f = \frac{a (1 - e^2)}{1 + e \cos f} \ \delta f \tag{G-64}$$

From Sections (G. 3), (G. 4), and (G. 5), the complete solution in terms of the variables f and M is

$$\delta r = k_2 \cos f + k_3 \sin f + k_4 \left[- \frac{3 e M \sin f}{2 (1 - e^2)^{3/2}} + \frac{1}{1 + e \cos f} \right] \tag{G-65}$$

$$\delta s = \frac{k_1 \, a \, (1 - e^2)}{1 + e \cos f} - \frac{k_2 \, (2 + e \cos f) \sin f}{1 + e \cos f}$$

$$+ \frac{k_3 \, (2 + e \cos f) \cos f}{1 + e \cos f} - \frac{3 \, k_4 \, M \, (1 + e \cos f)}{2 \, (1 - e^2)^{3/2}} \qquad \text{(G-66)}$$

$$\delta z = \frac{k_5 \sin f}{1 + e \cos f} + \frac{k_6 \cos f}{1 + e \cos f} \qquad \text{(G-67)}$$

The variant motion in the z direction is an undamped oscillation whose period is equal to the period of the reference trajectory.

The variant motion in the reference trajectory plane is more easily analyzed if the equation for δs is re-arranged as follows:

$$\delta s = \frac{k_1 \, a \, (1 - e^2)}{1 + e \cos f} - k_2 \left(1 + \frac{1}{1 + e \cos f}\right) \sin f$$

$$+ k_3 \left(1 + \frac{1}{1 + e \cos f}\right) \cos f - \frac{3 \, k_4 \, M \, (1 + e \cos f)}{2 \, (1 - e^2)^{3/2}} \qquad \text{(G-68)}$$

In addition to an undamped oscillation whose period is equal to that of the reference trajectory, the variant motion in the reference trajectory plane contains an oscillation that is modulated by a ramp function. Thus, the motion in the reference plane is dynamically unstable; the amplitude of the variation in position increases steadily as the number of periods is increased.

It should be pointed out that this analysis is based on linear perturbation theory; the conclusions drawn are applicable only as long as the position variations from the reference trajectory are small.

APPENDIX H

DETERMINATION OF VARIANT MOTION FROM FIRST VARIATIONS OF ORBITAL ELEMENTS

H. 1 Summary

First variations are taken of the six orbital elements that define the motion along an elliptical reference trajectory. The motion along the actual trajectory is a function of these six variations and the known characteristics of the reference trajectory. The basic analysis is applicable to ellipses of low eccentricity (approximately circular) as well as ellipses of moderate eccentricity; it is not applicable when e is equal to unity.

The general equations are applied to the particular case when e is not very close to either zero or unity. It is shown that the resulting equations for position variation are analogous to those developed in Appendix G.

H. 2 Introduction

In the variant two-body problem, if the reference trajectory is known to be an ellipse of moderate eccentricity, and if there are no disturbing forces, then the actual trajectory, which is assumed to differ only slightly from the reference trajectory, must also be an ellipse of moderate eccentricity. One method of attacking the variant problem is to assume small variations in each of the six known orbital elements of the reference trajectory and to determine the effect of these variations on position as a function of time. It is convenient to use, instead of position on the actual trajectory, the difference between position on the actual trajectory at time t and position on the reference trajectory at the same time. This difference, in vector form, is $\delta \underline{r}$.

This approach to the problem is primarily geometric; it depends on the a priori assumption that the variant trajectory is an ellipse.

In contrast, the approach of Appendix G is analytic; it requires no such assumption. Indeed, the solution of Appendix G, with its secular term, hardly resembles any of the more familiar forms of the equations of elliptical motion.

H.3 Effect of Variation in Euler Angles

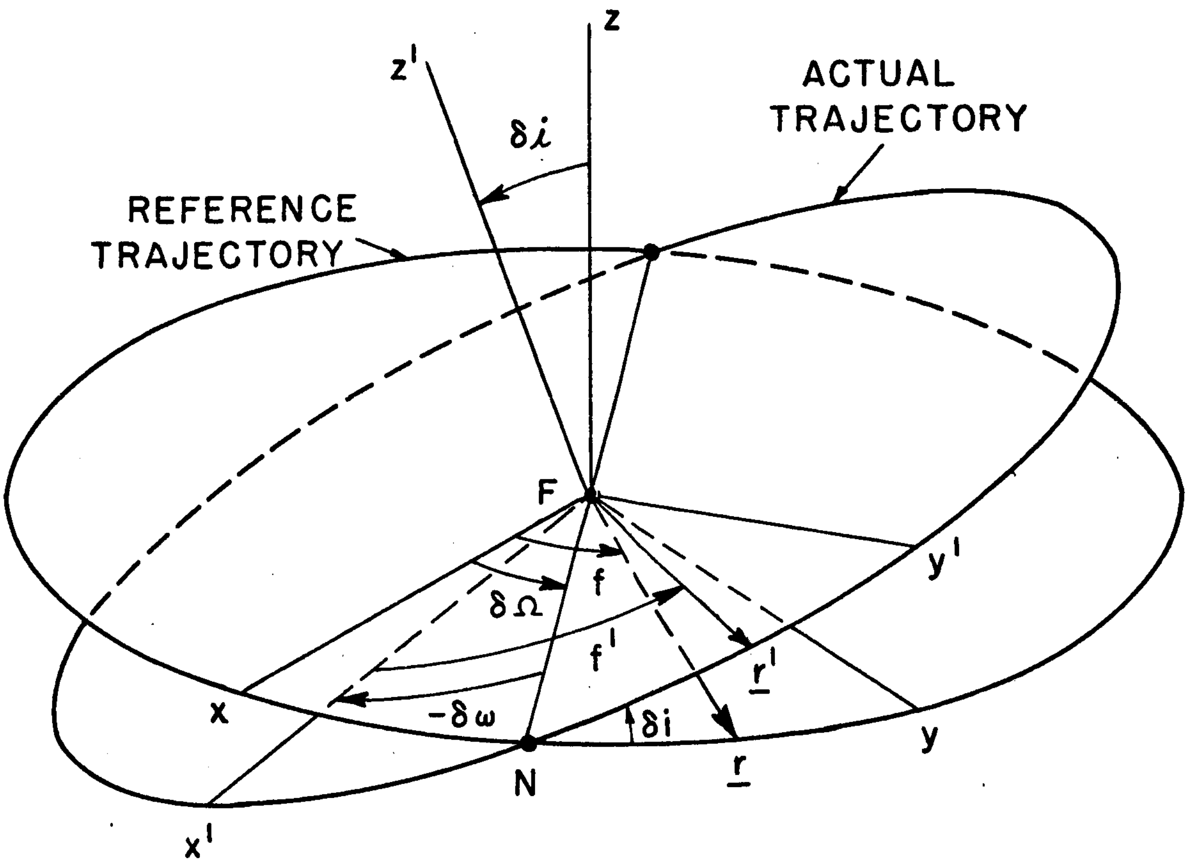
The x y z coordinate system, as defined in Appendix A, is related to the vehicle's reference trajectory plane. A new coordinate system, designated x' y' z', will now be introduced, with the axes of the new system bearing the same relationship to the actual trajectory that the axes of x y z bear to the reference trajectory. The origin of the new system is at the center of the sun. The x' - y' plane is the plane of the actual two-body trajectory. The positive x'-axis lies in the direction of perihelion from the sun. The positive y'-axis is 90° ahead of the positive x'-axis in the direction of vehicle motion. The positive z'-axis is parallel to the angular momentum vector of the actual trajectory. The x' y' z' coordinate system, like the x y z system, is a non-rotating coordinate system.

The Euler angles defining the orientation of the x' y' z' system relative to the x y z system are $\delta\Omega$, δi, and $\delta\omega$, as shown in Fig. H.1. Each of the three angles is regarded as a variation from its reference value, which is zero in each case. If the launch guidance were perfect, the x' y' z' and x y z systems would coincide.

The prime notation is used to designate characteristics of the actual trajectory. Thus, $\underline{r}'$ is the position vector on the actual trajectory, and f' is the true anomaly on the actual trajectory.

The vector $\underline{r}'$ can be resolved into its components along the r, s, and z axes of the reference trajectory local vertical coordinate system. The symbols used for the components are r'_r, r'_s, and r'_z. From Fig. H.1,

$$r'_r = r' \left[\cos(f' + \delta\omega) \cos(f - \delta\Omega) + \sin(f' + \delta\omega) \cos\delta i \sin(f - \delta\Omega) \right] \qquad \text{(H-1)}$$



F – focus at center of sun

FN – line of nodes

Fx,Fy,Fz – axes of reference trajectory stationary coordinate system

Fx',Fy',Fz' – axes of actual trajectory stationary coordinate system

$\delta\Omega, \delta i, \delta\omega$ – orientation angles between two coordinate systems

$\underline{r}$ – position vector on reference trajectory at time t

$\underline{r}'$ – position vector on actual trajectory at time t

f – true anomaly on reference trajectory at time t

f' – true anomaly on actual trajectory at time t

Figure H.1 Orientation of Actual Trajectory Relative to Reference Trajectory

$$r'_s = r' [-\cos(f' + \delta\omega) \sin(f - \delta\Omega)$$

$$+ \sin(f' + \delta\omega) \cos \delta i \cos(f - \delta\Omega)] \quad \text{(H-2)}$$

$$r'_z = r' \sin(f' + \delta\omega) \sin \delta i \quad \text{(H-3)}$$

The components of position variation vector $\delta \underline{r}$ along the r, s, and z axes are

$$\delta r = r'_r - r \quad \text{(H-4)}$$

$$\delta s = r'_s \quad \text{(H-5)}$$

$$\delta z = r'_z \quad \text{(H-6)}$$

The fundamental assumption of linear perturbation theory is that all variations from reference values be small. Thus, in Fig. H. 1, the separation of P' from P must be small, and, as a consequence, angle δi must be small. It is also necessary that the difference between $(f' + \delta\omega)$ and $(f - \delta\Omega)$ be small. This difference may be written as

$$(f' + \delta\omega) - (f - \delta\Omega) = \delta f + \delta\phi \quad \text{(H-7)}$$

where

$$\delta f = f' - f \quad \text{(H-8)}$$

$$\delta\phi = \delta(\omega + \Omega) \quad \text{(H-9)}$$

When the reference trajectory has appreciable eccentricity, it is necessary that the major axis of the actual trajectory be situated close to the major axis of the reference trajectory if P' is to be close to P for all values of f. Then, $\delta\phi$ must be small, and, since $(\delta f + \delta\phi)$ is always small, δf must likewise be small. It should be noted that the individual angles $\delta\phi$ and δf need not be small if the reference trajectory is circular or nearly circular, because for such trajectories a large displacement of the x'-axis from the x-axis has no appreciable effect, _per se_, on the distance of P' from P.

When the usual small-angle assumptions are applied to δi and $(\delta f + \delta\phi)$, the components of $\delta \underline{r}$ become

$$\delta r = r' \cos(\delta f + \delta\phi) - r = r' - r \tag{H-10}$$

$$\delta s = r' \sin(\delta f + \delta\phi) = r'(\delta f + \delta\phi) \tag{H-11}$$

$$\delta z = r' \, \delta i \sin(f' + \delta\omega) \tag{H-12}$$

In the last equation, $(f' + \delta\omega)$ may be written as

$$\begin{aligned} f' + \delta\omega &= (f + \delta f) + (\delta\phi - \delta\Omega) \\ &= (f - \delta\Omega) + (\delta f + \delta\phi) \end{aligned} \tag{H-13}$$

Since δr is a small quantity, linear theory permits the following additional simplification of Eqs. (H-11) and (H-12).

$$\delta s = (r + \delta r)(\delta f + \delta\phi) = r(\delta f + \delta\phi) \tag{H-14}$$

$$\begin{aligned} \delta z &= (r + \delta r)\, \delta i \sin[(f - \delta\Omega) + (\delta f + \delta\phi)] \\ &= r \, \delta i \sin(f - \delta\Omega) \end{aligned} \tag{H-15}$$

Equations (H-10), (H-14), and (H-15) show the effects of variations in the Euler angles on the components of $\delta \underline{r}$. The radial component δr is unaffected. The transverse component δs varies linearly with $\delta\phi$. The orthogonal component δz depends upon both δi and $\delta\Omega$.

H.4 Variation in Eccentric Anomaly

As an intermediate step in the determination of δr and δs, it is useful to derive an expression for δE, the variation in the eccentric anomaly, in terms of variations in the orbital elements.

The discussion in the last section concerning the angle $(\delta f + \delta\phi)$ is applicable to both $(\delta E + \delta\phi)$ and $(\delta M + \delta\phi)$; i.e., $(\delta E + \delta\phi)$ and $(\delta M + \delta\phi)$ are small angles regardless of the eccentricity of the reference ellipse; if the eccentricity of the reference ellipse is not near zero, δE and δM are individually small, but they need not be if the eccentricity is near zero. These considerations also apply to δM_0, the variation in the mean anomaly at epoch. To preserve generality, the angles δE, δM, and δM_0 will not be assumed to be small in the initial development. Then

$$E' = E + \delta E = (E - \delta\phi) + (\delta E + \delta\phi) \tag{H-16}$$

$$M' = (M - \delta\phi) + (\delta M + \delta\phi) \tag{H-17}$$

$$M_o{}' = (M_0 - \delta\phi) + (\delta M_0 + \delta\phi) \tag{H-18}$$

From Eqs. (B-47) and (B-55),

$$M = n\,t + M_0 = E - e \sin E \tag{H-19}$$

For the actual orbit, at time t,

$$M' = (n + \delta n)\,t + M_o{}' = E' - (e + \delta e) \sin E' \tag{H-20}$$

(H-19) is subtracted from (H-20).

$$\begin{aligned}(\delta M + \delta\phi) &= t\,\delta n + (\delta M_0 + \delta\phi) \\ &= (\delta E + \delta\phi) - (e + \delta e) \sin E' + e \sin E\end{aligned} \tag{H-21}$$

The variation δn may be expressed in terms of δa by the use of (B-62).

$$\delta\mu = 0 = \delta(n^2 a^3) = 2\,n\,a^3\,\delta n + 3\,n^2 a^2\,\delta a \tag{H-22}$$

$$\delta n = -\frac{3\,n}{2\,a}\,\delta a \tag{H-23}$$

Also,

$$\sin E' = \sin(E - \delta\phi) + (\delta E + \delta\phi)\cos(E - \delta\phi) \; . \tag{H-24}$$

(H-23) and (H-24) are substituted into (H-21), second-order terms are neglected, and the resulting equation is solved for $(\delta E + \delta\phi)$.

$$\delta E + \delta\phi = \frac{-\frac{3}{2}\, n\, t\, \frac{\delta a}{a} + (\delta M_0 + \delta\phi) + (e + \delta e)\, \sin(E - \delta\phi) - e \sin E}{1 - e \cos(E - \delta\phi)} \tag{H-25}$$

H.5 General Equations for Components of Position Variation

Equation (B-49) is used to determine δr.

$$r = a(1 - e \cos E) \tag{H-26}$$

On the actual trajectory,

$$\begin{aligned} r' &= (a + \delta a)\,[1 - (e + \delta e) \cos E'] \\ &= (a + \delta a)\left\{1 - (e + \delta e)\,[\cos(E - \delta\phi) - (\delta E + \delta\phi) \sin(E - \delta\phi)]\right\} \\ &= a\,[1 - (e + \delta e) \cos(E - \delta\phi) + e(\delta E + \delta\phi) \sin(E - \delta\phi)] \\ &\quad + [1 - e \cos(E - \delta\phi)]\,\delta a \end{aligned} \tag{H-27}$$

$$\begin{aligned} \delta r = r' - r &= a\,[e \cos E - (e + \delta e) \cos(E - \delta\phi) \\ &\quad + e(\delta E + \delta\phi) \sin(E - \delta\phi)] + [1 - e \cos(E - \delta\phi)]\,\delta a \end{aligned} \tag{H-28}$$

From (H-14), the deviation in the transverse direction is

$$\begin{aligned} \delta s &= r(\delta f + \delta\phi) = r'(\delta f + \delta\phi) \\ &= r' \sin(\delta f + \delta\phi) = r' \sin[f' - (f - \delta\phi)] \\ &= r'\,[\sin f' \cos(f - \delta\phi) - \cos f' \sin(f - \delta\phi)] \end{aligned} \tag{H-29}$$

From (B-53),

$$\sin f' = \frac{[1 - (e + \delta e)^2]^{1/2} \sin E'}{1 - (e + \delta e) \cos E'}$$

$$= \frac{(a + \delta a)}{r'} [1 - (e + \delta e)^2]^{1/2} [\sin (E - \delta\phi) + (\delta E + \delta\phi) \cos (E - \delta\phi)] \tag{H-30}$$

From (B-54),

$$\cos f' = \frac{\cos E' - (e + \delta e)}{1 - (e + \delta e) \cos E'}$$

$$= \frac{(a + \delta a)}{r'} [\cos (E - \delta\phi) - (\delta E + \delta\phi) \sin (E - \delta\phi) - (e + \delta e)] \tag{H-31}$$

(H-30) and (H-31) are substituted into (H-29). When higher-order terms are neglected, the expression for δs is

$$\begin{aligned} \delta s = a \Big\{ & (1 - e^2 - 2 e\, \delta e)^{1/2} \sin (E - \delta\phi) \cos (f - \delta\phi) \\ & - \big[\cos (E - \delta\phi) - (e + \delta e)\big] \sin (f - \delta\phi) \\ & + \big[(1 - e^2)^{1/2} \cos (E - \delta\phi) \cos (f - \delta\phi) \\ & + \sin (E - \delta\phi) \sin (f - \delta\phi)\big] (\delta E + \delta\phi) \Big\} \end{aligned} \tag{H-32}$$

Equations (H-28) and (H-32) are the general equations for δr and δs, applicable over a wide range of eccentricities for ellipses, from $e = 0$ to e approaching unity as a limit. The equations are not applicable when the reference ellipse is rectilinear (that is, when e is equal to one), for in that case a positive variation in e causes the actual trajectory to become hyperbolic.

(H-28) and (H-32) are used in conjunction with (H-25) to express δr and δs in terms of variations in the elements a, e, M_0, and ϕ. δz is independent of variations in these elements.

H. 6 Position Deviation for Trajectories of Moderate Eccentricity

Elliptical reference trajectories for which e is not very small (close to zero) or very large (close to unity) may be referred to as trajectories of "moderate" eccentricity. Practical trajectories for voyages to neighboring planets fall into this category. The general equations of Sections H. 4 and H. 5 will be used to obtain simplified expressions for δr and δs when the eccentricity is moderate.

Whenever the eccentricity is appreciably greater than zero, if the position variations are to remain small, δf, δE, δM, δM_0, and $\delta\phi$ must be small angles. Then Eq. (H-25) becomes

$$(\delta E + \delta\phi)(1 - e \cos E - e\,\delta\phi \sin E)$$

$$\doteq -\frac{3}{2} n t \frac{\delta a}{a} + \delta M_0 + \delta\phi - e\,\delta\phi \cos E + \sin E\,\delta e \tag{H-33}$$

$$\delta E = \frac{-\frac{3}{2} n t \frac{\delta a}{a} + \delta M_0 + \sin E\,\delta e}{1 - e \cos E} \tag{H-34}$$

From (H-28), the equation for δr is

$$\delta r = a\,[-e \sin E\,\delta\phi - \cos E\,\delta e + e \sin E\,(\delta E + \delta\phi)]$$
$$+ (1 - e \cos E)\,\delta a \tag{H-35}$$

(H-34) is substituted into (H-35)

$$\delta r = \left[a(1 - e \cos E) - \frac{\frac{3}{2} n a e \sin E}{1 - e \cos E} t \right] \frac{\delta a}{a}$$

$$+ \frac{a e \sin E}{1 - e \cos E}\,\delta M_0 - \frac{a(\cos E - e)}{1 - e \cos E}\,\delta e \tag{H-36}$$

With the use of the relations of Appendix B, (H-36) reduces to

$$\delta r = \left(r - \frac{3}{2} v_r t\right) \frac{\delta a}{a} + \frac{v_r}{n} \delta M_0 - a \cos f \, \delta e \qquad \text{(H-37)}$$

The derivation for δs proceeds in a similar fashion from Eqs. (H-32) and (H-34) and the standard forms of Appendix B. Note that

$$(1 - e^2 - 2e\,\delta e)^{1/2} = \left[(1 - e^2)\left(1 - \frac{2e\,\delta e}{1 - e^2}\right)\right]^{1/2}$$

$$= (1 - e^2)^{1/2} - \frac{e}{(1 - e^2)^{1/2}} \delta e \qquad \text{(H-38)}$$

Then δs is obtained as follows:

$$\delta s = a \left\{ \left[(1 - e^2)^{1/2} - \frac{e}{(1 - e^2)^{1/2}} \delta e \right] (\sin E - \cos E\, \delta\phi)(\cos f + \sin f\, \delta\phi) \right.$$

$$- (\cos E - e + \sin E\, \delta\phi - \delta e)(\sin f - \cos f\, \delta\phi)$$

$$\left. + [(1 - e^2)^{1/2} \cos E \cos f + \sin E \sin f](\delta E + \delta\phi) \right\}$$

$$= a \left\{ \left[-\frac{e}{(1 - e^2)^{1/2}} \sin E \cos f + \sin f \right] \delta e \right.$$

$$+ [(1 - e^2)^{1/2} (\sin E \sin f - \cos E \cos f) + (\cos E - e) \cos f$$

$$\left. - \sin E \sin f]\, \delta\phi + (1 - e^2)^{1/2} (\delta E + \delta\phi) \right\}$$

$$= a \left[\frac{\sin f}{1 + e \cos f} \delta e + (1 - e \cos E)\, \delta\phi + (1 - e^2)^{1/2} \delta E \right]$$

$$= \frac{-\frac{3}{2} n a (1 - e^2)^{1/2}}{1 - e \cos E} t \frac{\delta a}{a} + \frac{a (1 - e^2)^{1/2}}{1 - e \cos E} \delta M_0$$

$$+ a \left(\frac{2 + e \cos f}{1 + e \cos f}\right) \sin f\, \delta e + a (1 - e \cos E)\, \delta\phi$$

$$= -\frac{3}{2} v_s t \frac{\delta a}{a} + \frac{v_s}{n} \delta M_0 + \left(a + \frac{r}{1 - e^2}\right) \sin f\, \delta e + r\, \delta\phi \qquad \text{(H-39)}$$

H.7 Relation between Solution of Appendix G and Solution of Appendix H

Since the component equations of Appendix G, specifically (G-65), (G-66), and (G-67), are written in terms of the variables f and M, Eqs. (H-37), (H-39), and (H-15) of this appendix will be written in terms of the same variables so that the two solutions can be compared.

From (H-37),

$$\delta r = \left[\frac{a\,(1 - e^2)}{1 + e \cos f} - \frac{\frac{3}{2}\, n\, a\, e \sin f}{(1 - e^2)^{1/2}}\, t \right] \frac{\delta a}{a}$$

$$+ \frac{a\, e \sin f}{(1 - e^2)^{1/2}}\, \delta M_0 - a \cos f\, \delta e$$

$$= a\,(1 - e^2) \left[- \frac{3\, M\, e \sin f}{2\,(1 - e^2)^{3/2}} + \frac{1}{1 + e \cos f} \right] \frac{\delta a}{a}$$

$$+ \frac{a\, e}{(1 - e^2)^{1/2}} \sin f \left[\delta M_0 + \frac{3}{2}\, M_0\, \frac{\delta a}{a} \right] - a \cos f\, \delta e \qquad \text{(H-40)}$$

The expression $[\,\delta M_0 + 3/2\, M_0\, \delta a/a]$ may be simplified.

$$\delta M_0 = \delta(-\, n\, t_o) = -\, n\, \delta t_0 - t_0\, \delta n$$

$$= -\, n\, \delta t_0 - \frac{3}{2}\, M_0\, \frac{\delta a}{a} \qquad \text{(H-41)}$$

$$\delta M_0 + \frac{3}{2}\, M_0\, \frac{\delta a}{a} = -\, n\, \delta t_0 \qquad \text{(H-42)}$$

Finally,

$$\delta r = a\,(1 - e^2) \left[- \frac{3\, M\, e \sin f}{2\,(1 - e^2)^{3/2}} + \frac{1}{1 + e \cos f} \right] \frac{\delta a}{a}$$

$$- \frac{n\, a\, e}{(1 - e^2)^{1/2}} \sin f\, \delta t_0 - a \cos f\, \delta e \qquad \text{(H-43)}$$

From (H-39),

$$\delta s = -\frac{3\,n\,a\,(1 + e\cos f)}{2\,(1 - e^2)^{1/2}}\; t\; \frac{\delta a}{a} + \frac{a\,(1 + e\cos f)}{(1 - e^2)^{1/2}}\,\delta M_0$$

$$+ a\left(\frac{2 + e\cos f}{1 + e\cos f}\right)\sin f\,\delta e + \frac{a\,(1 - e^2)}{1 + e\cos f}\,\delta\phi$$

$$= a\,(1 - e^2)\left[-\frac{3\,M\,(1 + e\cos f)}{2\,(1 - e^2)^{3/2}}\right]\frac{\delta a}{a}$$

$$-\frac{n\,a\,e}{(1 - e^2)^{1/2}}\left(\frac{2 + e\cos f}{1 + e\cos f}\right)\cos f\,\delta t_0$$

$$+ a\left(\frac{2 + e\cos f}{1 + e\cos f}\right)\sin f\,\delta e$$

$$+ \frac{a\,(1 - e^2)}{1 + e\cos f}\left[\delta\phi - \frac{n\,\delta t_0}{(1 - e^2)^{3/2}}\right] \qquad \text{(H-44)}$$

From (H-15),

$$\delta z = \frac{a\,(1 - e^2)}{1 + e\cos f}\,\delta i\,(\sin f\cos\delta\Omega - \cos f\sin\delta\Omega) \qquad \text{(H-45)}$$

When (H-43), (H-44), and (H-45) are compared with (G-65), (G-66), and (G-67), respectively, it is evident that the two sets of equations are identical if

$$k_1 = \delta\phi - \frac{n\,\delta t_0}{(1 - e^2)^{3/2}} \qquad \text{(H-46)}$$

$$k_2 = -\,a\,\delta e \qquad \text{(H-47)}$$

$$k_3 = -\frac{n\,a\,e}{(1 - e^2)^{1/2}}\,\delta t_0 \qquad \text{(H-48)}$$

$$k_4 = a(1 - e^2) \frac{\delta a}{a} \tag{H-49}$$

$$k_5 = a(1 - e^2) \delta i \cos \delta\Omega \tag{H-50}$$

$$k_6 = - a(1 - e^2) \delta i \sin \delta\Omega \tag{H-51}$$

Thus, despite the presence of the secular term, the motion described by Eqs. (G-65), (G-66), and (G-67) is elliptical motion. The secular term is simply a manifestation of the fact that the period of the actual elliptical trajectory differs slightly from the period of the reference trajectory.

APPENDIX I

VARIATION IN POSITION, VELOCITY, AND ACCELERATION

I.1 Summary

The equations for position variation and velocity variation are expressed in vector form and also in matrix form in the three reference trajectory coordinate systems. An expression developed for variation in acceleration serves as a check of the basic solution of the variant equations of motion.

I.2 Vector Forms

It is evident from Fig. A.2 that components along the x, y axes and p, q axes may be derived from the r, s components by means of the following coordinate transformations:

$$\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \begin{pmatrix} \cos f & -\sin f \\ \sin f & \cos f \end{pmatrix} \begin{pmatrix} \delta r \\ \delta s \end{pmatrix} \tag{I-1}$$

$$\begin{pmatrix} \delta p \\ \delta q \end{pmatrix} = \begin{pmatrix} \cos \gamma & -\sin \gamma \\ \sin \gamma & \cos \gamma \end{pmatrix} \begin{pmatrix} \delta r \\ \delta s \end{pmatrix} \tag{I-2}$$

With the aid of these transformations and the relations of Sections B.8 and B.9, Eqs. (H-15), (H-37), and (H-39) may be combined into a single vector equation.

$$\delta \underline{r} = v\left(\frac{\delta M_0}{n} - \frac{3}{2} t \frac{\delta a}{a}\right) \underline{u}_q + r \frac{\delta a}{a} \underline{u}_r$$

$$+ \left(\frac{y}{1 - e^2} \delta e + r \delta \phi\right) \underline{u}_s - a \delta e \underline{u}_x$$

$$+ r \sin (f - \delta\Omega) \delta i \underline{u}_z \qquad \text{(I-3)}$$

The velocity deviation vector $\delta \underline{v}$ is obtained by vector differentiation of $\delta \underline{r}$. The angular velocity of the p, q axes is $\dot{g}$, the angular velocity of the r, s axes is $\dot{f}$, and the x, y, z axes are non-rotating.

$$\delta \underline{v} = - v \dot{g} \left(\frac{\delta M_0}{n} - \frac{3}{2} t \frac{\delta a}{a}\right) \underline{u}_p$$

$$+ [\dot{v} \frac{\delta M_0}{n} - \frac{3}{2} (\dot{v} t + v) \frac{\delta a}{a}] \underline{u}_q$$

$$+ \left[\dot{r} \frac{\delta a}{a} - \dot{f} \left(\frac{y}{1 - e^2} \delta e + r \delta \phi\right)\right] \underline{u}_r$$

$$+ \left(r \dot{f} \frac{\delta a}{a} + \frac{\dot{y}}{1 - e^2} \delta e + \dot{r} \delta \phi\right) \underline{u}_s$$

$$+ [\dot{r} \sin (f - \delta\Omega) + r \dot{f} \cos (f - \delta\Omega)] \delta i \underline{u}_z \qquad \text{(I-4)}$$

The terms in (I-4) may be simplified when the proper substitutions are made for $\dot{f}$, $\dot{\gamma}$, $\dot{r}$, and $\dot{y}$ from the relations of Appendix B.

The coefficient of $\left(\frac{\delta M_0}{n} - \frac{3}{2} t \frac{\delta a}{a}\right)$ in (I-4) is

$$- v \dot{g} \underline{u}_p + \dot{v} \underline{u}_q = (a_p \underline{u}_p + a_q \underline{u}_q) = a_r \underline{u}_r \tag{I-5}$$

The additional terms involving $\frac{\delta a}{a}$ are

$$-\frac{3}{2} v \underline{u}_q + \dot{r} \underline{u}_r + r \dot{f} \underline{u}_s = -\frac{v}{2} \underline{u}_q \tag{I-6}$$

The coefficient of $\frac{\delta e}{1 - e^2}$ is

$$- y \dot{f} \underline{u}_r + \dot{y} \underline{u}_s = - r \dot{f} \sin f \underline{u}_r + (\dot{r} \sin f + r \dot{f} \cos f) \underline{u}_s$$

$$= - v \sin f \underline{u}_p + v_s \cos f \underline{u}_s \tag{I-7}$$

The coefficient of $\delta \phi$ is

$$- r \dot{f} \underline{u}_r + \dot{r} \underline{u}_s = - v \underline{u}_p \tag{I-8}$$

The coefficient of $\delta i \, \underline{u}_z$ is

$$\dot{r} \sin(f - \delta\Omega) + r\dot{f}\cos(f - \delta\Omega)$$

$$= v[\sin\gamma \sin(f - \delta\Omega) + \cos\gamma\cos(f - \delta\Omega)]$$

$$= v\cos(g - \delta\Omega) \qquad \text{(I-9)}$$

With these substituions, Eq. (I-4) becomes

$$\delta\underline{v} = -v\left(\frac{\sin f}{1 - e^2}\,\delta e + \delta\phi\right)\underline{u}_p$$

$$-\frac{v}{2}\frac{\delta a}{a}\,\underline{u}_q + a_r\left(\frac{\delta M_0}{n} - \frac{3}{2}\,t\,\frac{\delta a}{a}\right)\underline{u}_r$$

$$+\frac{v_s\cos f}{1 - e^2}\,\delta e\,\underline{u}_s + v\cos(g - \delta\Omega)\,\delta i\,\underline{u}_z \qquad \text{(I-10)}$$

I. 3 Component Equations in Matrix Form

The component equations for $\delta\underline{r}$ and $\delta\underline{v}$ in the three reference trajectory coordinate systems are obtained from (I-3) and (I-10). Equations (I-11), (I-12), and (I-13) relate position variation and velocity variation in the reference trajectory plane to variations in the elements a, M_0, e, and ϕ. Equation (I-14) relates δz and δv_z to variations in i and Ω.

$$\begin{pmatrix} \delta x \\ \delta y \\ \delta v_x \\ \delta v_y \end{pmatrix} = \begin{pmatrix} x - \frac{3}{2} v_x t & \frac{v_x}{n} & -a - \frac{y \sin f}{1-e^2} & -y \\ y - \frac{3}{2} v_y t & \frac{v_y}{n} & \frac{y \cos f}{1-e^2} & x \\ -\frac{v_x}{2} - \frac{3}{2} a_x t & \frac{a_x}{n} & -\frac{(v_y + v_s \cos f)\sin f}{1-e^2} & -v_y \\ -\frac{v_y}{2} - \frac{3}{2} a_y t & \frac{a_y}{n} & \frac{v_x \sin f + v_s \cos^2 f}{1-e^2} & v_x \end{pmatrix} \begin{pmatrix} \frac{\delta a}{a} \\ \delta M_0 \\ \delta e \\ \delta \phi \end{pmatrix} \tag{I-11}$$

$$\begin{pmatrix} \delta r \\ \delta s \\ \delta v_r \\ \delta v_s \end{pmatrix} = \begin{pmatrix} r - \frac{3}{2} v_r t & \frac{v_r}{n} & -a \cos f & 0 \\ -\frac{3}{2} v_s t & \frac{v_s}{n} & a \sin f + \frac{y}{1-e^2} & r \\ -\frac{v_r}{2} - \frac{3}{2} a_r t & \frac{a_r}{n} & -\frac{v_s \sin f}{1-e^2} & -v_s \\ -\frac{v_s}{2} & 0 & \frac{v_y}{1-e^2} & v_r \end{pmatrix} \begin{pmatrix} \frac{\delta a}{a} \\ \delta M_0 \\ \delta e \\ \delta \phi \end{pmatrix} \tag{I-12}$$

$$
\begin{pmatrix} \delta p \\ \delta q \\ \delta v_p \\ \delta v_q \end{pmatrix}
=
\begin{pmatrix}
p & 0 & - a \cos g - \dfrac{y \sin \gamma}{1 - e^2} & -q \\
q - \dfrac{3}{2} v t & \dfrac{v}{n} & 2 a \sin g & p \\
- \dfrac{3}{2} a_p t & \dfrac{a_p}{n} & - \dfrac{v \sin f + v_s \cos f \sin \gamma}{1 - e^2} & -v \\
- \dfrac{v}{2} - \dfrac{3}{2} a_q t & \dfrac{a_q}{n} & \dfrac{v_s \cos f \cos \gamma}{1 - e^2} & 0
\end{pmatrix}
\begin{pmatrix} \dfrac{\delta a}{a} \\ \delta M_0 \\ \delta e \\ \delta \phi \end{pmatrix}
\tag{I-13}
$$

$$\begin{pmatrix} \delta z \\ \\ \delta v_z \end{pmatrix} = \begin{pmatrix} y & & -x \\ \\ v_y & & -v_x \end{pmatrix} \begin{pmatrix} \delta i \cos \delta\Omega \\ \\ \delta i \sin \delta\Omega \end{pmatrix} \tag{I-14}$$

I.4 Variation in Acceleration

The variation in acceleration may be obtained by vector differentiation of (I-10), and the result can be used to check the solution obtained for the matrix differential equation

$$\delta \underline{a} = \overset{*}{G}\, \delta \underline{r} \tag{I-15}$$

The result of differentiating (I-10) is

$$\delta \underline{a} = \left[-\dot{v}\left(\frac{\sin f}{1 - e^2}\ \delta e + \delta \phi\right) - \frac{v\, \dot{f} \cos f}{1 - e^2}\ \delta e + \frac{v\dot{g}}{2}\ \frac{\delta a}{a}\right] \underline{u}_p$$

$$+ \left[-v\dot{g}\left(\frac{\sin f}{1 - e^2}\ \delta e + \delta \phi\right) - \frac{\dot{v}}{2}\ \frac{\delta a}{a} \right] \underline{u}_q$$

$$+ \left[\dot{a}_r \left(\frac{\delta M_0}{n} - \frac{3}{2}\, t\, \frac{\delta a}{a}\right) - \frac{3}{2}\, a_r \frac{\delta a}{a} - \frac{v_s\, \dot{f} \cos f}{1 - e^2}\ \delta e \right] \underline{u}_r$$

$$+ \left[a_r\, \dot{f} \left(\frac{\delta M_0}{n} - \frac{3}{2}\, t\, \frac{\delta a}{a}\right) + \frac{\dot{v}_s \cos f - v_s\, \dot{f} \sin f}{1 - e^2}\ \delta e \right] \underline{u}_s$$

$$+ \left[\dot{v} \cos (g - \delta\Omega) - v\dot{g} \sin (g - \delta\Omega)\right]\ \delta i\ \underline{u}_z \tag{I-16}$$

The coefficient of $\delta\,\phi$ in (I-16) is

$$- (\dot{v}\,\underline{u}_p + v\,\dot{g}\,\underline{u}_q) = - (a_q\,\underline{u}_p - a_p\,\underline{u}_q)$$

$$= - a_r\,(\sin\gamma\,\underline{u}_p - \cos\gamma\,\underline{u}_q) = a_r\,\underline{u}_s \qquad \text{(I-17)}$$

The coefficient of $\frac{1}{2}\;\frac{\delta a}{a}$ is

$$v\,\dot{g}\,\underline{u}_p - \dot{v}\,\underline{u}_q = - a_p\,\underline{u}_p - a_q\,\underline{u}_q = - a_r\,\underline{u}_r \qquad \text{(I-18)}$$

The coefficient of $\left(\frac{\delta M_0}{n} - \frac{3}{2}\,t\,\frac{\delta a}{a}\right)$ is

$$\dot{a}_r\,\underline{u}_r + a_r\,\dot{f}\,\underline{u}_s = \frac{\mu}{r^3}\,(2\,\dot{r}\,\underline{u}_r - r\,\dot{f}\,\underline{u}_s)$$

$$= \frac{\mu}{r^3}\,(2\,v_r\,\underline{u}_r - v_s\,\underline{u}_s) \qquad \text{(I-19)}$$

The complete coefficient of $\frac{\delta a}{a}$ is

$$\frac{v\dot{g}}{2}\,\underline{u}_p - \frac{\dot{v}}{2}\,\underline{u}_q - \frac{3}{2}\,(\dot{a}_r\,t + a_r)\,\underline{u}_r - \frac{3}{2}\,a_r\,\dot{f}\,t\,\underline{u}_s$$

$$= - \frac{1}{2}\,a_r\,\underline{u}_r - \frac{3}{2}\,t\,\frac{\mu}{r^3}\,(2\,v_r\,\underline{u}_r - v_s\,\underline{u}_s) - \frac{3}{2}\,a_r\,\underline{u}_r$$

$$= \frac{\mu}{r^3}\,\left[\,(2\,r - 3\,v_r\,t)\,\underline{u}_r + \frac{3}{2}\,v_s\,t\,\underline{u}_s\,\right] \qquad \text{(I-20)}$$

One of the terms in the coefficient of δe contains the derivative $\dot{v}_s$. A substitution may be made for $\dot{v}_s$ by utilizing the fact that the angular momentum of the reference trajectory is constant.

$$h = r^2 \dot{f} = r\, v_s \tag{I-21}$$

$$\dot{h} = \dot{r}\, v_s + r\, \dot{v}_s = 0 \tag{I-22}$$

$$\dot{v}_s = - \frac{\dot{r}\, v_s}{r} = - v_r\, \dot{f} \tag{I-23}$$

The complete coefficient of $\dfrac{\delta e}{1 - e^2}$ is

$$(-\dot{v} \sin f - v\, \dot{f} \cos f)\, \underline{u}_p - v\, \dot{g} \sin f\, \underline{u}_q - v_s\, \dot{f} \cos f\, \underline{u}_r$$

$$+ (\dot{v}_s \cos f - v_s\, \dot{f} \sin f)\, \underline{u}_s$$

$$= a_r \sin f\, \underline{u}_s - v\, \dot{f} \cos f\, \underline{u}_p - v_s\, \dot{f} \cos f\, \underline{u}_r$$

$$- \dot{f}\, (v_r \cos f + v_s \sin f)\, \underline{u}_s \tag{I-24}$$

The product $v_s\, \dot{f}$ may be expanded as follows:

$$v_s\, \dot{f} = r\, \dot{f}^2 = \frac{h^2}{r^3} = \frac{h^2}{\mu}\, \frac{\mu}{r^3} = \frac{\mu\, a\, (1 - e^2)}{r^3} \tag{I-25}$$

With the aid of (I-25), the coefficient of $\frac{\delta e}{1 - e^2}$ becomes

$$- 2 v_s \dot{f} \cos f \underline{u}_r + (a_r - v_s \dot{f}) \sin f \underline{u}_s$$

$$= \frac{\mu}{r^3} \left\{ - 2 a (1 - e^2) \cos f \underline{u}_r + [- y - a (1 - e^2) \sin f] \underline{u}_s \right\} \tag{I-26}$$

The coefficient of $\delta i \underline{u}_z$ in (I-16) is

$$\dot{v} \cos (g - \delta \Omega) - v \dot{g} \sin (g - \delta\Omega)$$

$$= a_r [\sin \gamma \cos (g - \delta \Omega) + \cos \gamma \sin (g - \delta \Omega)]$$

$$= a_r \sin (f - \delta \Omega) \tag{I-27}$$

The relations for the terms comprising $\delta \underline{a}$, as expressed in Equations (I-17) through (I-27), contain components only in the r, s, and z directions. On the basis of these relations a matrix equation can now be written for $\delta \underline{a}$ in terms of the variations in the orbital elements.

$$\begin{pmatrix} \delta a_r \\ \delta a_s \\ \delta a_z \end{pmatrix} = \frac{\mu}{r^3} \begin{pmatrix} 2r - 3v_r t & \frac{2v_r}{n} & -2a\cos f & 0 & 0 & 0 \\ \frac{3}{2} v_s t & -\frac{v_s}{n} & -a\sin f - \frac{y}{1-e^2} & -r & 0 & 0 \\ 0 & 0 & 0 & 0 & -y & x \end{pmatrix} \begin{pmatrix} \frac{\delta a}{a} \\ \delta M_0 \\ \delta e \\ \delta \phi \\ \delta i \cos \delta \Omega \\ \delta i \sin \delta \Omega \end{pmatrix} \tag{I-28}$$

When Eq. (I-28) is compared with Eqs. (I-12) and (I-14), it may be seen that

$$\begin{pmatrix} \delta a_r \\ \delta a_s \\ \delta a_z \end{pmatrix} = \frac{\mu}{r^3} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \delta r \\ \delta s \\ \delta z \end{pmatrix} \tag{I-29}$$

The 3-by-3 diagonal matrix on the right-hand side of (I-29) is identical with the matrix on the right-hand side of (G-1). This is the $\overset{*}{G}$ matrix in the r s z coordinate system. Thus the solution for $\delta \underline{r}$ given by (I-3) has been checked.

APPENDIX J
LOW-ECCENTRICITY REFERENCE TRAJECTORIES

J.1 Summary

Equations are developed for position variation and velocity variation in low-eccentricity reference orbits. The differential equation solution of Appendix G is shown to be applicable to low-eccentricity orbits as well as orbits of moderate eccentricity.

J.2 Introduction

Although low-eccentricity trajectories cannot be used as transfer orbits on interplanetary voyages, the variant equations for such trajectories are derived in this appendix in order to illustrate the applicability of the general equations developed in Sections H.4 and H.5. The results obtained are of value in preliminary qualitative studies of the motion of satellites in circular or near-circular orbits.

J.3 Position Variation and Velocity Variation

In order to distinguish the results of this appendix from those of previous appendices, the subscript o will be added to all designations for orbital elements.

The distinctive feature of the reference orbits now being considered is that the eccentricity e is of the same order of magnitude as the orbital element variations. This characteristic is used in deriving expressions for δr and δs from Eqs. (H-25), (H-28), and (H-32.)

For low-eccentricity orbits,

$$1 - e_o \cos(E - \delta\phi_o) = 1 \tag{J-1}$$

$$r = a_o \tag{J-2}$$

$$\sin E = \frac{(1 - e_o^2)^{1/2} \sin f}{1 + e_o \cos f} = \sin f \tag{J-3}$$

$$\cos E = \frac{\cos f + e_o}{1 + e_o \cos f} = \cos f + e_o \tag{J-4}$$

$$v = \frac{n_o a_o (1 + e_o \cos E)^{1/2}}{(1 - e_o \cos E)^{1/2}} = n_o a_o \tag{J-5}$$

$$v_r = \frac{n_o a_o e_o \sin f}{(1 - e_o^2)^{1/2}} = n_o a_o e_o \sin f \tag{J-6}$$

$$v_s = \frac{n_o a_o (1 + e_o \cos f)}{(1 - e_o^2)^{1/2}} = n_o a_o = v \tag{J-7}$$

In Eq. (H-25) the term $(e_o + \delta e_o) \sin (E - \delta\phi_o)$ may be expanded as

$$(e_o + \delta e_o) \sin (E - \delta\phi_o) = (e_o + \delta e_o)(\sin f \cos \delta\phi_o - \cos f \sin \delta\phi_o) \tag{J-8}$$

The angle $(\delta E + \delta\phi_o)$, which must be small, is

$$\delta E + \delta\phi_o = -\frac{3}{2} n_o t \frac{\delta a_o}{a_o} + (\delta M_{oo} + \delta\phi_o)$$

$$- \sin f \left[e_o - (e_o + \delta e_o) \cos \delta\phi_o\right] - \cos f \,(e_o + \delta e_o) \sin \delta\phi_o \tag{J-9}$$

In Eq. (H-28), the term $e_o\,(\delta E + \delta\phi_o)$ is of second order in the small quantities. Therefore, the expression for δr is simply,

$$\delta r = a_o \left\{\frac{\delta a_o}{a_o} + \cos f \left[e_o - (e_o + \delta e_o) \cos \delta\phi_o\right]\right.$$

$$\left. - \sin f \,(e_o + \delta e_o) \sin \delta\phi_o\right\} \tag{J-10}$$

From Eq. (H-32),

$$\delta s = a_o \left[\sin (E - f) + (e_o + \delta e_o) \sin (f - \delta\phi_o)\right.$$

$$\left. + (\delta E + \delta\phi_o) \cos (E - f)\right] \tag{J-11}$$

$$\sin (E - f) = \sin E \cos f - \sin f \cos E = - e_o \sin f \tag{J-12}$$

$$\cos (E - f) = \cos E \cos f + \sin E \sin f = 1 + e_o \cos f = 1 \tag{J-13}$$

Equations (J-9), (J-12), and (J-13) are substituted into (J-11).

$$\delta s = a_o \left\{ -\frac{3}{2} n_o t \frac{\delta a_o}{a_o} + (\delta M_{oo} + \delta\phi_o) - 2 \sin f \left[e_o - (e_o + \delta e_o) \cos \delta\phi_o \right] - 2 \cos f \, (e_o + \delta e_o) \sin \delta\phi_o \right\} \tag{J-14}$$

From Eq. (H-15),

$$\delta z = a_o \, \delta i_o \sin (f - \delta\Omega_o) \tag{J-15}$$

The velocity deviation components are obtained by differentiating the components of $\delta \underline{r}$, with consideration being given to the fact that the coordinate system is rotating with angular velocity $\dot{f}$.

$$f = \frac{v_s}{r} = n_o \tag{J-16}$$

$$\delta v_r = v \left\{ \frac{3}{2} n_o t \frac{\delta a_o}{a_o} - (\delta M_{oo} + \delta\phi_o) + \sin f \left[e_o - (e_o + \delta e_o) \cos \delta\phi_o) \right] + \cos f \, (e_o + \delta e_o) \sin \delta\phi_o \right\} \tag{J-17}$$

$$\delta v_s = v \left\{ -\frac{1}{2} \frac{\delta a_o}{a_o} - \cos f \left[e_o - (e_o + \delta e_o) \cos \delta\phi_o \right] + \sin f \, (e_o + \delta e_o) \sin \delta\phi_o \right\} \tag{J-18}$$

$$\delta v_z = v\ \delta i_o \cos (f - \delta\Omega_o) \tag{J-19}$$

The position and velocity deviations may be written in matrix form as shown in Eq. (J-20).

J.4 Variation in Acceleration

As in Appendix I, the variation in acceleration may be used to check the solution of the variant problem. The matrix for $\delta \underline{a}$ is obtained by differentiating the lower half of (J-20). Equations (J-21) and (J-22) indicate that the solution checks satisfactorily.

J.5 Comparison with Differential Equation Solution of Appendix G

When the eccentricity is small, the differential equation solution given by Eqs. (G-65), (G-66), and (G-67) reduces to the following:

$$\delta r = k_2 \cos f + k_3 \sin f + k_4 \tag{J-23}$$

$$\delta s = k_1\, a_o - 2\, k_2 \sin f + 2\, k_3 \cos f - \frac{3}{2}\, k_4\, M \tag{J-24}$$

$$\delta z = k_5 \sin f + k_6 \cos f \tag{J-25}$$

A comparison of Eqs. (J-23), (J-24), and (J-25) with the first three equations of (J-20) indicates that the two sets are identical if

$$k_1 = \delta\phi_o - n_o\ \delta t_{oo} \tag{J-26}$$

$$k_2 = a_o \left[e_o - (e_o + \delta e_o) \cos\ \delta\phi_o \right] \tag{J-27}$$

$$k_3 = -\, a_o\, (e_o + \delta e_o) \sin \delta\phi_o \tag{J-28}$$

$$k_4 = \delta a_o \tag{J-29}$$

$$k_5 = a_o\ \delta i_o \cos \delta\Omega_o \tag{J-30}$$

$$\begin{pmatrix} \frac{\delta r}{a_o} \\ \frac{\delta s}{a_o} \\ \frac{\delta z}{a_o} \\ \frac{\delta v_r}{v} \\ \frac{\delta v_s}{v} \\ \frac{\delta v_z}{v} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cos f & -\sin f & 0 & 0 \\ -\frac{3}{2} n_o t & 1 & -2 \sin f & -2 \cos f & 0 & 0 \\ 0 & 0 & 0 & 0 & \sin f & -\cos f \\ \frac{3}{2} n_o t & -1 & \sin f & \cos f & 0 & 0 \\ -\frac{1}{2} & 0 & -\cos f & \sin f & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos f & \sin f \end{pmatrix} \begin{pmatrix} \frac{\delta a_o}{a_o} \\ \delta M_{oo} + \delta\phi_o \\ e_o - (e_o + \delta e_o) \cos \delta\phi_o \\ (e_o + \delta e_o) \sin \delta\phi_o \\ \delta i_o \cos \delta\Omega_o \\ \delta i_o \sin \delta\Omega_o \end{pmatrix} \tag{J-20}$$

$$\begin{pmatrix} \delta a_r \\ \delta a_s \\ \delta a_z \end{pmatrix} = n_o^2 a_o \begin{pmatrix} 2 & 0 & 2\cos f & -2\sin f & 0 & 0 \\ \frac{3}{2} n_o t & -1 & 2\sin f & 2\cos f & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sin f & \cos f \end{pmatrix} \begin{pmatrix} \frac{\delta a_o}{a_o} \\ \delta M_{oo} + \delta\phi_o \\ e_o - (e_o + \delta e_o)\cos\delta\phi_o \\ (e_o + \delta e_o)\sin\delta\phi_o \\ \delta i_o \cos\delta\Omega_o \\ \delta i_o \sin\delta\Omega_o \end{pmatrix} \tag{J-21}$$

$$= \frac{\mu}{a_o^3} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \delta r \\ \delta s \\ \delta z \end{pmatrix} \tag{J-22}$$

$$k_6 = - a_o \ \delta i_o \sin \delta\Omega_o \tag{J-31}$$

Thus, the differential equation solution is applicable to low-eccentricity reference orbits as well as reference orbits of moderate eccentricity. The distinction between the two types of reference orbits reduces simply to a difference in the physical interpretation of the six constants of integration.

APPENDIX K
MATRICES FOR ELLIPTICAL TRAJECTORIES

K.1 Summary

For the case when the reference trajectory is an ellipse, analytic expressions are developed for the elements of the matrices defined in Appendix F. The eccentric anomaly E is the independent variable. The reference trajectory flight path coordinate system is used.

K.2 Selection of a Coordinate System

The matrices associated with the problem of small departures from a known reference trajectory are defined in Appendix F. In Appendices G, H, and I, the variational problem is solved analytically for the case when the reference trajectory is an ellipse of moderate eccentricity.. The solution is an expression for position variation and velocity variation in terms of the variations in the orbital elements and the characteristics of the reference trajectory. From this basic solution analytic expressions can be derived for all the matrices of Appendix F.

The algebraic and trigonometric manipulations required are straightforward but quite formidable in length and in number. Therefore, the choice of coordinate system, of independent variable, and of a group of six orbital elements should be carefully considered from the standpoint of reducing as much as possible the amount of mathematical drudgery.

The reference trajectory coordinate systems have the obvious advantage of uncoupling the z-axis variant motion from the variant motion in the reference trajectory plane. The consequence of this uncoupling is that in each 3-by-3 matrix or sub-matrix of the group of matrices in Appendix F, at least four of the nine elements are zero.

The problem now is to select one of the three reference trajectory systems. Each of the three has an advantage not possessed by the other two. The x y z system is non-rotating, and hence the matrix $\overset{*}{W}$ in Appendix F is the zero matrix. In the r s z system, both the nominal position vector and the nominal acceleration vector lie in the r direction, so that there is no component of either vector in the s direction. The p q z system has the advantage that the nominal velocity vector is in the q direction; hence, there is no component of $\underline{v}$ in the p direction.

The matrix formulations (I-11), (I-12), and (I-13) may be used to compare the three systems. The 4-by-4 matrix of the x y z system has no zeros; the 4-by-4 matrix of the r s z system has two zeros, one due to the fact that $s = 0$ and the other due to the fact that $a_s = 0$; the p q z system's 4-by-4 matrix has two zeros, both due to the fact that $v_p = 0$. It is apparent that both the r s z and p q z systems are preferable to the x y z system.

The final choice between the r s z system and the p q z system is a difficult one. Actually, a considerable amount of analysis was done in each of the two systems before it became apparent that the matrix formulations are simpler in the p q z system. The relative simplicity of the p q z system is associated with the fact that in this system the secular term in position variation is wholly along the q-axis.

It might be argued that the r s z system has a similar property, in that the secular term in velocity variation is wholly along the r-axis, and therefore, analysis in the r s z system ought to be just as simple as analysis in the p q z system. This argument is not valid because one of the useful formulations in guidance theory involves expressing the variant path in terms of the three components of position variation at two different times, as illustrated by Eqs. (F-2), (F-8), and (F-19), and no such formulation in terms of the three components of velocity variation at two different times is required.

K. 3 Selection of an Independent Variable

The analysis is facilitated if all time-varying quantities are expressed in terms of one independent variable. Variables that might be used include time itself and the three anomalies f, E, and M.

Inasmuch as t and M are linearly related, the choice of one or the other of the two would appear to be equally desirable. Both have the decided disadvantage that trigonometric functions of E and f can be expressed in terms of M (or t) only through Kepler's equation, (B-55), which cannot be solved explicitly for E in terms of M.

On the other hand, the use of the true anomaly f as the independent variable causes difficulty when the secular term in the solution of the variant equations is expressed in terms of f.

By process of elimination, then, the eccentric anomaly is chosen as the independent variable. Both trigonometric and secular terms can be expressed directly in terms of E.

K. 4 Selection of a Grouping of Orbital Elements

The final selection problem is that of selecting a group of six independent constants which characterize the variant path. As in the case of choosing a coordinate system and an independent variable, the criterion in making the selection is to reduce the amount of algebra to manageable proportions.

The six constants serve as a bridge linking position and velocity variation at one time to position and velocity variation at another time. First, a 6-by-6 matrix is obtained which relates position and velocity variation at time t_j to the six constants; then the 6-by-6 matrix is inverted so that the six constants can be expressed in terms of the position and velocity variations at time t_i. Finally, the two 6-by-6 matrices, one in terms of t_j and the other in terms of t_i, are multiplied together to yield a single 6-by-6 matrix by means of which position and velocity variations at t_j may be expressed in terms of position and velocity variations at t_i. The final matrix is the transition matrix $\overset{*}{C}_{ji}$ of Appendix F.

The six constants may be conveniently expressed in terms of variations of some combination of the six orbital elements. The grouping that has finally been chosen, written in vector form, is the following:

$$\delta \underline{e} = \begin{Bmatrix} (1 - e^2)^{1/2} \; \delta\phi - n \; \delta t_o \\ \dfrac{\delta e}{(1 - e^2)^{1/2}} \\ \dfrac{1}{2} \; \dfrac{\delta a}{a} \\ e \; \delta\phi \\ (1 - e^2)^{1/2} \; \delta i \cos \delta\Omega \\ \delta i \sin \; \delta\Omega \end{Bmatrix} \qquad \text{(K-1)}$$

K. 5 The Use of Position Variation and Velocity Variation to Describe the Motion in the Reference Trajectory Plane

The first four elements of $\delta \underline{e}$ are related to the motion in the reference trajectory plane; the last two are related to the motion normal to the reference trajectory plane. Since the two types of motion are uncoupled, they can be studied independently. This section and the one immediately following will be devoted to a study of the motion in the reference trajectory plane.

If the elements in the vector on the right-hand side of (I-13) are replaced by the first four elements of (K-1), the equation may be rewritten in the form shown in (K-2). When the factor $1/(1 - e^2 \cos^2 E)^{1/2}$ is considered as part of the 4-by-4 matrix of (K-2), the determinant of the matrix is unity. Equation (K-3) is obtained by inverting (K-2). The dashed lines in (K-2) and (K-3) indicate matrix partitioning.

$$
\begin{Bmatrix} \dfrac{\delta p}{a} \\[2ex] \dfrac{\delta q}{a} \\[2ex] \dfrac{\delta v_p}{na} \\[2ex] \dfrac{\delta v_q}{na} \end{Bmatrix}
= \frac{1}{(1 - e^2 \cos^2 E)^{1/2}}
\left\{
\begin{array}{l}
(1 - e\cos E) \begin{pmatrix} 0 & -(\cos E + e) & 2(1 - e^2)^{1/2} & -\sin E \end{pmatrix} \\
\hdashline
\begin{pmatrix} 1 + e\cos E & 2(1 - e^2)^{1/2} \sin E & \begin{array}{l} -3(E - e\sin E)(1 + e\cos E) \\ +2e\sin E(1 - e\cos E) \end{array} & -2(1 - e^2)^{1/2} \cos E \end{pmatrix} \\
\hdashline
\dfrac{1}{(1 - e\cos E)^2} \begin{pmatrix} -(1 - e^2)^{1/2} & -\left[\begin{array}{l}(1 - e\cos E)e\cos E \\ +(1 - e^2)\end{array}\right]\sin E & 3(E - e\sin E)(1 - e^2)^{1/2} & \begin{array}{l}(1 - e\cos E)e\cos^2 E \\ +(\cos E - e)\end{array} \\[3ex] -e\sin E & (1 - e^2)^{1/2}(\cos E - e) & \begin{array}{l} 3(E - e\sin E)e\sin E \\ -(1 + e\cos E)(1 - e\cos E)^2 \end{array} & (1 - e^2)^{1/2}\sin E \end{pmatrix}
\end{array}
\right\}
\begin{Bmatrix} (1 - e^2)^{1/2}\ \delta\phi \\ -n\ \delta t_o \\[2ex] \dfrac{\delta e}{(1 - e^2)^{1/2}} \\[2ex] \dfrac{1}{2}\ \dfrac{\delta a}{a} \\[2ex] e\ \delta\phi \end{Bmatrix}
\qquad (K\text{-}2)
$$

$$
\begin{Bmatrix} (1 - e^2)^{1/2}\ \delta\phi \\ -n\ \delta t_o \\[2ex] \dfrac{\delta e}{(1 - e^2)^{1/2}} \\[2ex] \dfrac{1}{2}\ \dfrac{\delta a}{a} \\[2ex] e\ \delta\phi \end{Bmatrix}
= \frac{1}{(1 - e^2 \cos^2 E)^{1/2}}
\left\{
\frac{1}{(1 - e\cos E)^2}
\begin{pmatrix}
3(E - e\sin E)(1 - e^2)^{1/2} & \begin{array}{l} 3(E - e\sin E)e\sin E \\ -(1 + e\cos E)(1 - e\cos E)^2 \end{array} \\[3ex]
\begin{array}{l}(1 - e\cos E)e\cos^2 E \\ +(\cos E - e)\end{array} & (1 - e^2)^{1/2}\sin E \\[3ex]
(1 - e^2)^{1/2} & e\sin E \\[3ex]
\left[\begin{array}{l}(1 - e\cos E)e\cos E \\ +(1 - e^2)\end{array}\right]\sin E & -(1 - e^2)^{1/2}(\cos E - e)
\end{pmatrix}
\;\vdots\;
(1 - e\cos E)
\begin{pmatrix} -2(1 - e^2)^{1/2} \\[3ex] \sin E \\[3ex] 0 \\[3ex] -(\cos E + e) \end{pmatrix}
\;\vdots\;
\begin{pmatrix} \begin{array}{l} 3(E - e\sin E)(1 + e\cos E) \\ -2e\sin E(1 - e\cos E) \end{array} \\[3ex] 2(1 - e^2)^{1/2}\cos E \\[3ex] 1 + e\cos E \\[3ex] 2(1 - e^2)^{1/2}\sin E \end{pmatrix}
\right\}
\begin{Bmatrix} \dfrac{\delta p}{a} \\[2ex] \dfrac{\delta q}{a} \\[2ex] \dfrac{\delta v_p}{na} \\[2ex] \dfrac{\delta v_q}{na} \end{Bmatrix}
\qquad (K\text{-}3)
$$

There is a striking similarity between the elements of the 4-by-4 matrix of (K-2) and the elements of the 4-by-4 matrix of (K-3). The similarity is made more apparent by partitioning the matrix of (K-2) into four 2-by-2 matrices as follows:

$$\begin{Bmatrix} \frac{\delta p}{a} \\ \frac{\delta q}{a} \\ \frac{\delta v_p}{n\,a} \\ \frac{\delta v_q}{n\,a} \end{Bmatrix} = \begin{Bmatrix} \overset{*}{A}_1 & \overset{*}{A}_2 \\ \overset{*}{A}_3 & \overset{*}{A}_4 \end{Bmatrix} \begin{Bmatrix} (1-e^2)^{1/2}\,\delta\phi - n\,\delta t_o \\ \frac{\delta e}{(1-e^2)^{1/2}} \\ \frac{1}{2}\,\frac{\delta a}{a} \\ e\,\delta\phi \end{Bmatrix} \tag{K-4}$$

In terms of the four $\overset{*}{A}$ matrices, Eq. (K-3) becomes

$$\begin{Bmatrix} (1-e^2)^{1/2}\,\delta\phi - n\,\delta t_o \\ \frac{\delta e}{(1-e^2)^{1/2}} \\ \frac{1}{2}\,\frac{\delta a}{a} \\ e\,\delta\phi \end{Bmatrix} = \begin{Bmatrix} \overset{*}{A}_4{}^T & -\overset{*}{A}_2{}^T \\ -\overset{*}{A}_3{}^T & \overset{*}{A}_1{}^T \end{Bmatrix} \begin{Bmatrix} \frac{\delta p}{a} \\ \frac{\delta q}{a} \\ \frac{\delta v_p}{n\,a} \\ \frac{\delta v_q}{n\,a} \end{Bmatrix} \tag{K-5}$$

The terms of the 4-by-4 matrix of (K-5), which is the inverse of the 4-by-4 matrix of (K-4), can obviously be obtained from the matrix of (K-4) by inspection. This relationship between (K-4) and (K-5) is not true in general for any arbitrary selection of orbital element variations; in fact, the grouping of the elements that is being used has been chosen primarily because it validates the simple relation between (K-4) and (K-5).

When the subscript j is added to each of the $\overset{*}{A}$ matrices in (K-4) in order to indicate that the matrices are evaluated at time t_j, the equation gives the position and velocity variations corresponding to $t = t_j$. Similarly, adding the subscript i to the $\overset{*}{A}{}^T$ matrices of (K-5) signifies that the variations in the orbital elements are being expressed in terms of position and velocity variations at $t = t_i$. If the two resulting equations are combined, the variations in the elements may be eliminated and the position and velocity variations at $t = t_j$ are related to the position and velocity variations at $t = t_i$.

$$\left\{ \begin{matrix} \delta p_j \\ \delta q_j \\ \delta v_{p_j} \\ \delta v_{p_j} \end{matrix} \right\} = \left\{ \begin{matrix} \overset{*}{A}_{1j} & \overset{*}{A}_{2j} \\ n\overset{*}{A}_{3j} & n\overset{*}{A}_{4j} \end{matrix} \right\} \left\{ \begin{matrix} \overset{*}{A}_{4i}{}^T & -\frac{1}{n}\overset{*}{A}_{2i}{}^T \\ -\overset{*}{A}_{3i}{}^T & \frac{1}{n}\overset{*}{A}_{1i}{}^T \end{matrix} \right\} \left\{ \begin{matrix} \delta p_i \\ \delta q_i \\ \delta v_{p_i} \\ \delta v_{p_i} \end{matrix} \right\} \qquad \text{(K-6)}$$

$$\begin{Bmatrix} \delta p_j \\ \delta q_j \\ \delta v_{p_j} \\ \delta v_{q_j} \end{Bmatrix} = \begin{bmatrix} \overset{*}{A}_{1j} \overset{*}{A}_{4i}{}^T - \overset{*}{A}_{2j} \overset{*}{A}_{3i}{}^T & \frac{1}{n} (-\overset{*}{A}_{1j} \overset{*}{A}_{2i}{}^T + \overset{*}{A}_{2j} \overset{*}{A}_{1i}{}^T) \\ n (\overset{*}{A}_{3j} \overset{*}{A}_{4i}{}^T - \overset{*}{A}_{4j} \overset{*}{A}_{3i}{}^T) & -\overset{*}{A}_{3j} \overset{*}{A}_{2i}{}^T + \overset{*}{A}_{4j} \overset{*}{A}_{1i}{}^T \end{bmatrix} \begin{Bmatrix} \delta p_i \\ \delta q_i \\ \delta v_{p_i} \\ \delta v_{q_i} \end{Bmatrix} \tag{K-7}$$

$$= \begin{bmatrix} \overset{*}{M}_{ji}{}' & \overset{*}{N}_{ji}{}' \\ \overset{*}{S}_{ji}{}' & \overset{*}{T}_{ji}{}' \end{bmatrix} \begin{Bmatrix} \delta p_i \\ \delta q_i \\ \delta v_{p_i} \\ \delta v_{q_i} \end{Bmatrix} \tag{K-8}$$

The primed matrices of (K-8) are the two-dimensional versions of the corresponding matrices defined in Appendix F. It is apparent from (K-7) and (K-8) that

$$\overset{*}{M}_{ji}{}' = (\overset{*}{T}_{ij}{}')^T \tag{K-9}$$

$$\overset{*}{N}_{ji}{}' = (-\overset{*}{N}_{ij}{}')^T \tag{K-10}$$

$$\overset{*}{S}_{ji}{}' = (-\overset{*}{S}_{ij}{}')^T \tag{K-11}$$

$$\overset{*}{T}_{ji}{}' = (\overset{*}{M}_{ij}{}')^T \tag{K-12}$$

$$
\begin{Bmatrix} \dfrac{\delta p_i}{a} \\[2ex] \dfrac{\delta q_i}{a} \\[2ex] \dfrac{\delta p_j}{a} \\[2ex] \dfrac{\delta q_j}{a} \end{Bmatrix}
=
\begin{Bmatrix}
\dfrac{1 - e\cos E_i}{(1 - e^2\cos^2 E_i)^{1/2}} \begin{pmatrix} 0 & -(\cos E_i + e) & 2(1-e^2)^{1/2} & -\sin E_i \end{pmatrix} \\[2ex]
\dfrac{1}{(1 - e^2\cos^2 E_i)^{1/2}} \begin{pmatrix} 1 + e\cos E_i & 2(1-e^2)^{1/2}\sin E_i & \begin{array}{l} -(3E_i - e\sin E_i)(1 + e\cos E_i) \\ +4e\sin E_i \end{array} & -2(1-e^2)^{1/2}\cos E_i \end{pmatrix} \\[2ex]
\dfrac{1 - e\cos E_j}{(1 - e^2\cos^2 E_j)^{1/2}} \begin{pmatrix} 0 & -(\cos E_j + e) & 2(1-e^2)^{1/2} & -\sin E_j \end{pmatrix} \\[2ex]
\dfrac{1}{(1 - e^2\cos^2 E_j)^{1/2}} \begin{pmatrix} 1 + e\cos E_j & 2(1-e^2)^{1/2}\sin E_j & \begin{array}{l} -(3E_j - e\sin E_j)(1 + e\cos E_j) \\ +4e\sin E_j \end{array} & -2(1-e^2)^{1/2}\cos E_j \end{pmatrix}
\end{Bmatrix}
\begin{Bmatrix} (1-e^2)^{1/2}\,\delta\phi \\ -n\,\delta t_o \\[2ex] \dfrac{\delta e}{(1-e^2)^{1/2}} \\[2ex] \dfrac{1}{2}\,\dfrac{\delta a}{a} \\[2ex] e\,\delta\phi \end{Bmatrix}
\tag{K-13}
$$

These two-dimensional matrix equations are in agreement with the corresponding three-dimensional matrix equations of Section F. 7.

K. 6 The Use of Two Position Variations to Describe the Motion in the Reference Trajectory Plane

Another way of expressing the variations in the orbital elements is in terms of the position variations at two different times, t_i and t_j. This is accomplished by inverting Eq. (K-13). The expression for the inverse is simplified to some extent by the introduction of two new angles, E_P ("E plus") and E_M ("E minus").

$$E_P = \frac{1}{2} (E_j + E_i) \tag{K-14}$$

$$E_M = \frac{1}{2} (E_j - E_i) \tag{K-15}$$

The determinant of the 4-by-4 matrix of (K-13) is

$$(\det)_{pq} = - 4 X \sin E_M \tag{K-16}$$

where $X = (3 E_M - e \sin E_M \cos E_P)(\cos E_M + e \cos E_P) - 4 \sin E_M$ (K-17)

The inverse equation is (K-18).

(K-13) and (K-18) illustrate the reason previously mentioned for selecting the p q z coordinate system. Only two of the sixteen elements in the matrix of (K-13) contain the secular term; if the r s z system were used, there would be four elements with secular terms. Eight elements in the (K-18) matrix have secular terms; with the r s z system there would be twelve elements with secular terms.

Note that the element $\frac{1}{2} \frac{\delta a}{a}$ is unaffected by the secular term.

The factor 1/2 X is common to all the elements in the (K-18) matrix. Since X is a time-varying quantity that goes through zero, there are combinations of E_i and E_j for which the matrix of (K-13) become singular; for these combinations (K-18) cannot be evaluated.

$$
\begin{Bmatrix}
(1-e^2)^{1/2}\,\delta\phi \\ -n\,\delta t_o \\[2ex]
\dfrac{\delta e}{(1-e^2)^{1/2}} \\[2ex]
\dfrac{1}{2}\,\dfrac{\delta a}{a} \\[2ex]
e\,\delta\phi
\end{Bmatrix}
= \frac{1}{2X}
\left\{
\frac{1}{(1-e^2\cos^2 E_i)^{1/2}}
\begin{pmatrix}
\begin{aligned}
&2(1-e^2)^{1/2}\Big\{(1+e\cos E_i) \\
&\cdot\left[\frac{3E_M}{\sin E_M} - e\cos E_P\right] \\
&-(3E_j - e\sin E_j)\sin E_M \\
&-4\cos E_M\Big\}
\end{aligned} \\[2ex]
\begin{aligned}
&-(1+e\cos E_i)\sin E_j \\
&\cdot\left[\frac{3E_M}{\sin E_M} - e\cos E_P\right] \\
&+4(\sin E_P + e\sin E_M)
\end{aligned} \\[2ex]
-2(1-e^2)^{1/2}\sin E_M \\[2ex]
\begin{aligned}
&(1+e\cos E_i)(\cos E_j + e) \\
&\cdot\left[\frac{3E_M}{\sin E_M} - e\cos E_P\right] \\
&-4(\cos E_P + e\cos E_M)
\end{aligned}
\end{pmatrix}
\Bigg|\;
\frac{1-e\cos E_i}{(1-e^2\cos^2 E_i)^{1/2}}
\begin{pmatrix}
\begin{aligned}
&(3E_j - e\sin E_j) \\
&\cdot(\cos E_M + e\cos E_P) \\
&-4(\sin E_M + e\sin E_P)
\end{aligned} \\[2ex]
2(1-e^2)^{1/2}\cos E_P \\[2ex]
\cos E_M + e\cos E_P \\[2ex]
2(1-e^2)^{1/2}\sin E_P
\end{pmatrix}
\Bigg|\;
\frac{1}{(1-e^2\cos^2 E_j)^{1/2}}
\begin{pmatrix}
\begin{aligned}
&-2(1-e^2)^{1/2}\Big\{(1+e\cos E_j) \\
&\cdot\left[\frac{3E_M}{\sin E_M} - e\cos E_P\right] \\
&+(3E_i - e\sin E_i)\sin E_M \\
&-4\cos E_M\Big\}
\end{aligned} \\[2ex]
\begin{aligned}
&(1+e\cos E_j)\sin E_i \\
&\cdot\left[\frac{3E_M}{\sin E_M} - e\cos E_P\right] \\
&-4(\sin E_P - e\sin E_M)
\end{aligned} \\[2ex]
-2(1-e^2)^{1/2}\sin E_M \\[2ex]
\begin{aligned}
&-(1+e\cos E_j)(\cos E_i + e) \\
&\cdot\left[\frac{3E_M}{\sin E_M} - e\cos E_P\right] \\
&+4(\cos E_P + e\cos E_M)
\end{aligned}
\end{pmatrix}
\Bigg|\;
\frac{1-e\cos E_j}{(1-e^2\cos^2 E_j)^{1/2}}
\begin{pmatrix}
\begin{aligned}
&-(3E_i - e\sin E_i) \\
&\cdot(\cos E_M + e\cos E_P) \\
&-4(\sin E_M - e\sin E_P)
\end{aligned} \\[2ex]
-2(1-e^2)^{1/2}\cos E_P \\[2ex]
-(\cos E_M + e\cos E_P) \\[2ex]
-2(1-e^2)^{1/2}\sin E_P
\end{pmatrix}
\right\}
\begin{Bmatrix}
\dfrac{\delta p_i}{a} \\[2ex]
\dfrac{\delta q_i}{a} \\[2ex]
\dfrac{\delta p_j}{a} \\[2ex]
\dfrac{\delta q_j}{a}
\end{Bmatrix}
\qquad \text{(K-18)}
$$

The remarks about X are also applicable to sin E_M. Whenever sin E_M equals zero, the (K-13) matrix is singular. The significance of the singularities is discussed in Appendix O.

K. 7 Motion Normal to the Reference Trajectory Plane

The position and velocity variations along the z-axis are

$$\begin{Bmatrix} \delta z \\ \delta v_z \end{Bmatrix} = \begin{Bmatrix} y & -x \\ v_y & -v_x \end{Bmatrix} \begin{Bmatrix} \delta i \cos \delta\Omega \\ \delta i \sin \delta\Omega \end{Bmatrix} \qquad \text{(K-19)}$$

$$\begin{Bmatrix} \dfrac{\delta z}{a} \\ \dfrac{\delta v_z}{na} \end{Bmatrix} = \begin{Bmatrix} \sin E & -(\cos E - e) \\ \dfrac{\cos E}{1 - e \cos E} & \dfrac{\sin E}{1 - e \cos E} \end{Bmatrix} \begin{Bmatrix} (1 - e^2)^{1/2} \, \delta i \cos \delta\Omega \\ \delta i \sin \delta\Omega \end{Bmatrix} \qquad \text{(K-20)}$$

The determinant of the 2-by-2 matrix of (K-20) is equal to one. The inverse of (K-20) is

$$\begin{Bmatrix} (1 - e^2)^{1/2} \, \delta i \cos \delta\Omega \\ \delta i \sin \delta\Omega \end{Bmatrix} = \begin{Bmatrix} \dfrac{\sin E}{1 - e \cos E} & \cos E - e \\ \dfrac{-\cos E}{1 - e \cos E} & \sin E \end{Bmatrix} \begin{Bmatrix} \dfrac{\delta z}{a} \\ \dfrac{\delta v_z}{na} \end{Bmatrix} \qquad \text{(K-21)}$$

By combining (K-20) and (K-21), δz_j and δv_{z_j} are expressed in terms of δz_i and δv_{z_i}.

$$\begin{Bmatrix} \delta z_j \\ \delta v_{z_j} \end{Bmatrix} = \begin{Bmatrix} \sin E_j & -(\cos E_j - e) \\ \dfrac{n \cos E_j}{1 - e\cos E_j} & \dfrac{n \sin E_j}{1 - e \cos E_j} \end{Bmatrix} \begin{Bmatrix} \dfrac{\sin E_i}{1 - e\cos E_i} & \dfrac{1}{n}(\cos E_i - e) \\ -\dfrac{\cos E_i}{1 - e\cos E_i} & \dfrac{1}{n}\sin E_i \end{Bmatrix} \begin{Bmatrix} \delta z_i \\ \delta v_{z_i} \end{Bmatrix} \tag{K-22}$$

$$= \begin{Bmatrix} 1 - \dfrac{2\sin^2 E_M}{1 - e\cos E_i} & \dfrac{2\sin E_M (\cos E_M - e\cos E_P)}{n} \\ \dfrac{-2n \sin E_M \cos E_M}{(1 - e\cos E_i)(1 - e\cos E_j)} & 1 - \dfrac{2\sin^2 E_M}{1 - e\cos E_j} \end{Bmatrix} \begin{Bmatrix} \delta z_i \\ \delta v_{z_i} \end{Bmatrix} \tag{K-23}$$

Note that when $\sin E_M = 0$, i.e., when $(E_j - E_i)$ is an integer multiple of 360°, $\delta z_j = \delta z_i$ and $\delta v_{z_j} = \delta v_{z_i}$, irrespective of the nature of the variations in the orbital elements.

The variations in the elements may be expressed in terms of δz_i and δz_j by inverting Eq. (K-24).

$$\begin{Bmatrix} \dfrac{\delta z_i}{a} \\ \dfrac{\delta z_j}{a} \end{Bmatrix} = \begin{Bmatrix} \sin E_i & -(\cos E_i - e) \\ \sin E_j & -(\cos E_j - e) \end{Bmatrix} \begin{Bmatrix} (1 - e^2)^{1/2} & \delta i \cos & \delta\Omega \\ \delta i \sin \delta\Omega \end{Bmatrix} \tag{K-24}$$

The determinant of the 2-by-2 matrix is

$$(\det)_z = \sin(E_j - E_i) - e(\sin E_j - \sin E_i) \qquad \text{(K-25)}$$

$$= 2 \sin E_M (\cos E_M - e \cos E_P) \qquad \text{(K-26)}$$

The inverted equation is

$$\begin{pmatrix} (1-e^2)^{1/2} & \delta i \cos \delta\Omega \\ & \delta i \sin \delta\Omega \end{pmatrix} = \frac{1}{(\det)_z} \begin{pmatrix} -(\cos E_j - e) & \cos E_i - e \\ -\sin E_j & \sin E_i \end{pmatrix} \begin{pmatrix} \frac{\delta z_i}{a} \\ \frac{\delta z_j}{a} \end{pmatrix}$$

(K-27)

The condition for singularity of the matrix of (K-24) is most easily interpreted when $(\det)_z$ is expressed in terms of the difference in true anomalies, $(f_j - f_i.)$

$$(\det)_z = \frac{r_i \, r_j}{a^2 (1 - e^2)^{1/2}} \sin(f_j - f_i) \qquad \text{(K-28)}$$

The matrix becomes singular when

$$f_j - f_i = N\pi \qquad \text{(K-29)}$$

where N is any integer.

K.8 The Transition Matrix $\overset{*}{C}_{ji}$

The results of Section K. 5 and K. 7 can be combined to give analytic expressions for the matrices defined in Appendix F. In this section, such expressions are developed for the elements of the transition matrix $\overset{*}{C}_{ji}$.

The 3-by-6 matrices $\overset{*}{F}_j$ and $\overset{*}{L}_j$ satisfy the equation

$$\delta \underline{x}_j = \begin{Bmatrix} \delta p_j \\ \delta q_j \\ \delta z_j \\ \delta v_{p_j} \\ \delta v_{q_j} \\ \delta v_{z_j} \end{Bmatrix} = \begin{Bmatrix} \overset{*}{F}_j \\ \overset{*}{L}_j \end{Bmatrix} \begin{Bmatrix} (1-e^2)^{1/2} \; \delta\phi - n \;\; \delta t_o \\ \dfrac{\delta e}{(1-e^2)^{1/2}} \\ \dfrac{1}{2} \; \dfrac{\delta a}{a} \\ e \;\; \delta\phi \\ (1-e^2)^{1/2} \;\; \delta i \; \cos \;\; \delta\Omega \\ \delta i \; \sin \;\; \delta\Omega \end{Bmatrix} \tag{K-30}$$

The elements of $\overset{*}{F}_j$ and $\overset{*}{L}_j$ are given in Eqs. (K-31) and (K-32).

$$
\overset{*}{F}_j = a \left\{ \begin{array}{l}
\dfrac{1 - e\cos E_j}{(1 - e^2\cos^2 E_j)^{1/2}} \begin{pmatrix} 0 & -(\cos E_j + e) & 2(1-e^2)^{1/2} & -\sin E_j & 0 & 0 \end{pmatrix} \\
\hdashline
\dfrac{1}{(1 - e^2\cos^2 E_j)^{1/2}} \begin{pmatrix} 1 + e\cos E_j & 2(1-e^2)^{1/2}\sin E_j & \begin{matrix} -3(1+e\cos E_j)(E_j - e\sin E_j) \\ +2e\sin E_j(1 - e\cos E_j) \end{matrix} & -2(1-e^2)^{1/2}\cos E_j & 0 & 0 \end{pmatrix} \\
\hdashline
\begin{pmatrix} 0 & 0 & 0 & 0 & \sin E_j & -(\cos E_j - e) \end{pmatrix}
\end{array} \right\}
\tag{K-31}
$$

$$
\overset{*}{L}_j = na \left\{ \begin{array}{l}
\dfrac{1}{(1 - e\cos E_j)^2(1 - e^2\cos^2 E_j)^{1/2}} \begin{pmatrix} -(1-e^2)^{1/2} & -\left[(1 - e\cos E_j)e\cos E_j + (1-e^2)\right]\sin E_j & 3(1-e^2)^{1/2}(E_j - e\sin E_j) & \begin{matrix} (1 - e\cos E_j)e\cos^2 E_j \\ +(\cos E_j - e) \end{matrix} & 0 & 0 \\ -e\sin E_j & (1-e^2)^{1/2}(\cos E_j - e) & \begin{matrix} 3e\sin E_j(E_j - e\sin E_j) \\ -(1+e\cos E_j)(1-e\cos E_j)^2 \end{matrix} & (1-e^2)^{1/2}\sin E_j & 0 & 0 \end{pmatrix} \\
\hdashline
\dfrac{1}{1 - e\cos E_j} \begin{pmatrix} 0 & 0 & 0 & 0 & \cos E_j & \sin E_j \end{pmatrix}
\end{array} \right\}
\tag{K-32}
$$

$$
\overset{*}{R}_i = \frac{1}{a} \left\{ \begin{array}{c} \dfrac{1}{(1 - e\cos E_i)^2 (1 - e^2 \cos^2 E_i)^{1/2}} \begin{pmatrix} 3(1-e^2)^{1/2}(E_i - e\sin E_i) & \begin{array}{l} 3e\sin E_i\,(E_i - e\sin E_i) \\ -(1+e\cos E_i)(1-e\cos E_i)^2 \end{array} & 0 \\ \begin{array}{l} (1-e\cos E_i)e\cos^2 E_i \\ +(\cos E_i - e) \end{array} & (1-e^2)^{1/2}\sin E_i & 0 \\ (1-e^2)^{1/2} & e\sin E_i & 0 \\ \left[\begin{array}{l} (1-e\cos E_i)e\cos E_i \\ +(1-e^2) \end{array} \right] \sin E_i & -(1-e^2)^{1/2}(\cos E_i - e) & 0 \end{pmatrix} \\ \hline \dfrac{1}{(1-e\cos E_i)} \begin{pmatrix} 0 & 0 & \sin E_i \\ 0 & 0 & -\cos E_i \end{pmatrix} \end{array} \right\} \qquad \text{(K-34)}
$$

$$\overset{*}{V}_i = \frac{1}{na} \left\{ \frac{1 - e\cos E_i}{(1 - e^2\cos^2 E_i)^{1/2}} \begin{pmatrix} -2(1-e^2)^{1/2} \\ \sin E_i \\ 0 \\ -(\cos E_i + e) \\ 0 \\ 0 \end{pmatrix} \;\Bigg|\; \frac{1}{(1 - e^2\cos^2 E_i)^{1/2}} \begin{pmatrix} 3(1+e\cos E_i)(E_i - e\sin E_i) \\ \quad -2e\sin E_i(1 - e\cos E_i) \\ 2(1-e^2)^{1/2}\cos E_i \\ 1 + e\cos E_i \\ 2(1-e^2)^{1/2}\sin E_i \\ 0 \\ 0 \end{pmatrix} \;\Bigg|\; \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \cos E_i - e \\ \sin E_i \end{pmatrix} \right\} \qquad \text{(K-35)}$$

The inverse of Eq. (K-30) establishes the 6-by-3 matrices $\overset{*}{R}_i$ and $\overset{*}{V}_i$.

$$\delta\underline{e} = \begin{Bmatrix} (1-e^2)^{1/2} \; \delta\phi - n\;\delta t_o \\ \dfrac{\delta e}{(1-e^2)^{1/2}} \\ \dfrac{1}{2}\;\dfrac{\delta a}{a} \\ e\;\delta\phi \\ (1-e^2)^{1/2}\;\delta i \cos\;\delta\Omega \\ \delta i \sin\;\delta\Omega \end{Bmatrix} = \begin{Bmatrix} \overset{*}{R}_i & \overset{*}{V}_i \end{Bmatrix} \begin{Bmatrix} \delta p_i \\ \delta q_i \\ \delta z_i \\ \delta v_{p_i} \\ \delta v_{q_i} \\ \delta v_{z_i} \end{Bmatrix} \qquad \text{(K-33)}$$

The elements of $\overset{*}{R}_i$ and $\overset{*}{V}_i$ are given in Eqs. (K-34) and (K-35).

The transition matrix $\overset{*}{C}_{ji}$ is obtained from $\overset{*}{F}_j$, $\overset{*}{L}_j$, $\overset{*}{R}_i$, and $\overset{*}{V}_i$.

$$\overset{*}{C}_{ji} = \begin{Bmatrix} \overset{*}{M}_{ji} & \overset{*}{N}_{ji} \\ \overset{*}{S}_{ji} & \overset{*}{T}_{ji} \end{Bmatrix} \qquad \text{(K-36)}$$

$$= \begin{Bmatrix} \overset{*}{F}_j \\ \overset{*}{L}_j \end{Bmatrix} \begin{Bmatrix} \overset{*}{R}_i & \overset{*}{V}_i \end{Bmatrix} \qquad \text{(K-37)}$$

$$= \begin{Bmatrix} \overset{*}{F}_j\overset{*}{R}_i & \overset{*}{F}_j\overset{*}{V}_i \\ \overset{*}{L}_j\overset{*}{R}_i & \overset{*}{L}_j\overset{*}{V}_i \end{Bmatrix} \qquad \text{(K-38)}$$

$$
\overset{*}{M}_{ji} = \left\{ \begin{array}{c}
\dfrac{1}{(1-e^2\cos^2 E_i)^{1/2}(1-e^2\cos^2 E_j)^{1/2}}
\left(\begin{array}{lll}
\begin{array}{l}
(1+e\cos E_i)(1-e\cos E_j) \\
+\dfrac{2\sin E_M(1-e\cos E_j)}{(1-e\cos E_i)^2}\cdot\Big[(1-e^2)\sin E_M \\
-(1-e\cos E_i)e\sin E_i\cos E_M\Big]
\end{array}
&
\begin{array}{l}
\dfrac{2(1-e^2)^{1/2}(1-e\cos E_j)}{(1-e\cos E_i)^2} \\
\cdot\sin E_M(\cos E_M - e\cos E_P)
\end{array}
& 0 \\[3em]
\begin{array}{l}
\dfrac{2(1-e^2)^{1/2}}{(1-e\cos E_i)^2} \\
\cdot\Big\{-(1+e\cos E_j) \\
\cdot(3E_M - e\sin E_M\cos E_P) \\
+2\sin E_M\Big[e\cos E_P \\
+(1+e\cos E_i - e^2\cos^2 E_i)\cos E_M\Big]\Big\}
\end{array}
&
\begin{array}{l}
(1-e\cos E_i)(1+e\cos E_j) \\
+\dfrac{2}{(1-e\cos E_i)^2}\Big\{-(1+e\cos E_j) \\
\cdot e\sin E_i(3E_M - e\sin E_M\cos E_P) \\
+2\sin E_M\Big[2e\sin E_P \\
-(1+e^2)\sin E_M\Big]\Big\}
\end{array}
& 0
\end{array} \right) \\
\hline
\left(\begin{array}{ccc}
0 & 0 & 1-\dfrac{2\sin^2 E_M}{1-e\cos E_i}
\end{array} \right)
\end{array} \right\}
$$

(K-39)

$$\overset{*}{N}_{ji} = \frac{2}{n} \left\{ \begin{array}{c|c} \frac{1}{(1-e^2\cos^2 E_i)^{1/2}(1-e^2\cos E_j)^{1/2}} \begin{pmatrix} \begin{array}{l}(1-e\cos E_i)(1-e\cos E_j)\\ \cdot \sin E_M(\cos E_M + e\cos E_p)\end{array} & 2(1-e^2)^{1/2}(1-e\cos E_j)\sin^2 E_M \\ -2(1-e^2)^{1/2}(1-e\cos E_i)\sin^2 E_M & \begin{array}{l}-(1+e\cos E_i)(1+e\cos E_j)\\ \cdot(3E_M - e\sin E_M\cos E_P)\\ +4\sin E_M(\cos E_M + e\cos E_P)\end{array} \end{pmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline \begin{pmatrix} 0 & 0 \end{pmatrix} & \sin E_M(\cos E_M - e\cos E_P) \end{array} \right\} \qquad \text{(K-40)}$$

$$
S_{ji} = 2n \left\{ \begin{array}{c|c}
\dfrac{1}{(1-e\cos E_i)^2(1-e\cos E_j)^2(1-e^2\cos^2 E_i)^{1/2}(1-e^2\cos^2 E_j)^{1/2}} \left(\begin{array}{ccc}
\begin{array}{l}
(1-e^2)(3E_M - 2e\sin F_M \cos E_P) \\
-\Big\{(1-e\cos E_i)(1-e\cos E_j)e^2 \sin E_i \sin E_j \\
+(1-e^2)\left[1+(1-e\cos E_i)\, e\cos E_i\right. \\
\left.+(1-e\cos E_j)e\cos E_j\right]\Big\} \sin E_M \cos E_M
\end{array}
&
\begin{array}{l}
(1-e)^{1/2}\Big\{3e\sin F_i(E_M - e\sin E_M\cos E_P) \\
+\left[(1-e^2)+(1-e\cos E_j)e\cos E_j\right]\sin^2 E_M \\
\quad -e^2\left[2(1-e\cos E_i)\cos E_M\cos E_P\right. \\
\left.+(1-e\cos E_j)\cos E_j\right]\sin E_M\sin E_P\Big\}
\end{array}
& 0 \\[2em]
\begin{array}{l}
(1-e^2)^{1/2}\Big\{3e\sin E_j(E_M - e\sin E_M\cos E_P) \\
-\left[(1-e^2)+(1-e\cos E_i)e\cos E_i\right]\sin^2 E_M \\
-e^2\left[2(1-e\cos E_j)\cos E_M\cos E_P\right. \\
\left.+(1-e\cos E_i)\cos E_i\right]\sin E_M\sin E_P\Big\}
\end{array}
&
\begin{array}{l}
e^2\sin E_i\sin E_j\left[3E_M - 4e\sin E_M\cos E_P\right. \\
\left.+e^2\sin E_M\cos E_M(\cos^2 E_P - \sin^2 E_P)\right] \\
-(1-e^2)\sin E_M\left[(1+e^2)\cos E_M - 2e\cos E_P\right]
\end{array}
& 0
\end{array} \right) \\
\hline
\left(\begin{array}{ccc} 0 & 0 & -\dfrac{\sin E_M \cos E_M}{(1-e\cos E_i)(1-e\cos E_j)} \end{array} \right)
\end{array} \right\}
$$

(K-41)

$$
\overset{*}{T}_{ji} = \left\{ \begin{array}{c|c}
\dfrac{1}{(1-e^2\cos^2 E_i)^{1/2}(1-e^2\cos^2 E_j)^{1/2}} \left(\begin{array}{ccc}
\begin{array}{l}
(1+e\cos E_j)(1-e\cos E_i) \\
+\dfrac{2\sin E_M(1-e\cos E_i)}{(1-e\cos E_j)^2}\Big[(1-e^2)\sin E_M \\
+(1-e\cos E_j)\, e\sin E_j\cos E_M\Big]
\end{array}
&
\begin{array}{l}
\dfrac{2(1-e^2)^{1/2}}{(1-e\cos E_j)^2} \\
\cdot\Big\{(1+e\cos E_i) \\
\cdot(3E_M - e\sin E_M\cos E_P) \\
-2\sin E_M\Big[e\cos F_P \\
+(1+e\cos F_j - e^2\cos^2 E_j)\cos F_M\Big]\Big\}
\end{array}
& 0 \\
\\
\begin{array}{l}
-\dfrac{2(1-e^2)^{1/2}(1-e\cos E_i)}{(1-e\cos E_j)^2} \\
\cdot\sin E_M(\cos E_M - e\cos F_P)
\end{array}
&
\begin{array}{l}
(1-e\cos E_j)(1+e\cos E_i) \\
+\dfrac{2}{(1-e\cos E_j)^2}\Big\{(1+e\cos E_i) \\
\cdot e\sin E_j(3E_M - e\sin E_M\cos E_P) \\
-2\sin F_M\Big[2e\sin E_P \\
+(1+e^2)\sin E_M\Big]\Big\}
\end{array}
& 0
\end{array} \right) & \\
\hline
\left(\begin{array}{ccc} 0 & 0 & 1-\dfrac{2\sin^2 E_M}{1-e\cos E_j} \end{array} \right) &
\end{array} \right\}
$$

(K-42)

The matrix multiplications indicated in (K-38) have been performed. The four resulting 3-by-3 sub-matrices, $\overset{*}{M}_{ji}$, $\overset{*}{N}_{ji}$, $\overset{*}{S}_{ji}$ and $\overset{*}{T}_{ji}$, are presented in Eqs. (K-39), (K-40), (K-41), and (K-42). Taken together, these four sub-matrices constitute the desired solution for $\overset{*}{C}_{ji}$.

It is interesting to note that only nine of the sixteen non-zero in-plane elements of $\overset{*}{C}_{ji}$ have secular terms. There is no secular term in the expression for δp_j. The coefficient of δv_{p_i} contains no secular term.

K. 9 Matrices Associated with Position Variations at Two Different Times

The matrices of Appendix F that are used in conjunction with a path deviation vector composed of two position variation vectors are $\overset{*}{H}_{ij}$, $\overset{*}{H}_{ji}$, $\overset{*}{J}_{ij}$, and $\overset{*}{K}_{ij}$. The first two are defined by the following equation:

$$\begin{Bmatrix} (1-e^2)^{1/2} \; \delta\phi - n \; \delta t_o \\ \dfrac{\delta e}{(1-e^2)^{1/2}} \\ \dfrac{1}{2} \dfrac{\delta a}{a} \\ e \; \delta\phi \\ (1-e^2)^{1/2} \; \delta i \cos \; \delta\Omega \\ \delta i \sin \; \delta\Omega \end{Bmatrix} = \begin{Bmatrix} \overset{*}{H}_{ij} & \overset{*}{H}_{ji} \end{Bmatrix} \begin{Bmatrix} \delta p_i \\ \delta q_i \\ \delta z_i \\ \delta p_j \\ \delta q_j \\ \delta z_j \end{Bmatrix} \tag{K-43}$$

$$
\overset{*}{H}_{ij} = \frac{1}{2a}\left\{
\frac{1}{(1-e^2\cos^2 E_i)^{1/2}\,X}
\begin{pmatrix}
2(1-e^2)^{1/2}\left\{(1+e\cos E_i)\right. \\
\cdot\left(\dfrac{3E_M}{\sin E_M} - e\cos E_P\right) \\
-(3E_j - e\sin E_j)\sin E_M \\
\left. -4\cos E_M\right\} \\[1em]
-(1+e\cos E_i)\sin E_j \\
\cdot\left(\dfrac{3E_M}{\sin E_M} - e\cos E_P\right) \\
+4(\sin E_P + e\sin E_M) \\
-2(1-e^2)^{1/2}\sin E_M \\[1em]
(1+e\cos E_i)(\cos E_j + e) \\
\cdot\left(\dfrac{3E_M}{\sin E_M} - e\cos E_P\right) \\
-4(\cos E_P + e\cos E_M) \\[1em]
0 \\[1em]
0
\end{pmatrix}
\;\Bigg|\;
\frac{1-e\cos E_i}{(1-e^2\cos^2 E_i)^{1/2}\,X}
\begin{pmatrix}
(3E_j - e\sin E_j) \\
\cdot(\cos E_M + e\cos E_P) \\
-4(\sin E_M + e\sin E_P) \\[1em]
2(1-e^2)^{1/2}\cos E_P \\[1em]
\cos E_M + e\cos E_P \\[1em]
2(1-e^2)^{1/2}\sin E_P \\[1em]
0 \\[1em]
0
\end{pmatrix}
\;\Bigg|\;
\frac{1}{\sin E_M(\cos E_M - e\cos E_P)}
\begin{pmatrix}
0 \\[1em]
0 \\[1em]
0 \\[1em]
0 \\[1em]
-(\cos E_j - e) \\[1em]
-\sin E_j
\end{pmatrix}
\right\}
\qquad \text{(K-44)}
$$

$$
J^*_{ij} = n \left\{ \begin{array}{c} \dfrac{1}{1 + e\cos E_i} \begin{pmatrix} \begin{array}{l} \dfrac{e \sin E_i}{(1 - e\cos E_i)^2} \\ + \dfrac{1}{2(1 - e\cos E_i)\,X} \Bigg\{ (1 + e\cos E_i) \\ \cdot \left[2\sin^2 E_M - (1 + e\cos E_i) \right] \\ \cdot \left[\dfrac{3\,E_M}{\sin E_M} - e\cos E_P \right] \\ + 4\Big[(1 + e\cos E_i)\cos E_M \\ + e \sin E_i \sin E_M \Big] \Bigg\} \end{array} & -(1 - e^2)^{1/2} \left[\dfrac{1}{(1 - e\cos E_i)^2} + \dfrac{\sin E_M}{X} \right] & 0 \\ -(1 - e^2)^{1/2} \left[\dfrac{1}{(1 - e\cos E_i)^2} + \dfrac{\sin E_M}{X} \right] & \begin{array}{l} - \dfrac{e \sin E_i}{(1 - e\cos E_i)^2} \\ + \dfrac{1}{2X} (1 - e\cos E_i)(\cos E_M + e\cos E_P) \end{array} & 0 \end{pmatrix} \\ \hdashline \begin{pmatrix} 0 & 0 & \dfrac{2\sin^2 E_M - (1 - e\cos E_i)}{2\sin E_M (\cos E_M - e\cos E_P)(1 - e\cos E_i)} \end{pmatrix} \end{array} \right\}
$$

(K-47)

$$
\overset{*}{K}_{ij} = \frac{n}{2}
\left\{
\begin{array}{c}
\dfrac{1}{(1-e^2\cos^2 E_i)^{1/2}(1-e^2\cos^2 E_j)^{1/2}} \times
\begin{pmatrix}
\begin{array}{l}(1+e\cos E_i)(1+e\cos E_j) \\ \cdot\left(\dfrac{3\,E_M}{\sin E_M} - e\cos E_P\right) \\ -\,4\,(\cos E_M + e\cos E_P)\end{array} & 2\,(1-e^2)^{1/2}(1-e\cos E_j)\sin E_M & 0 \\
-2\,(1-e^2)^{1/2}(1-e\cos E_i)\sin E_M & \begin{array}{l}-(1-e\cos E_i)(1-e\cos E_j) \\ \cdot(\cos E_M + e\cos E_P)\end{array} & 0
\end{pmatrix} \\
\hdashline
\begin{pmatrix}
0 & 0 & \dfrac{1}{\sin E_M(\cos E_M - e\cos E_P)}
\end{pmatrix}
\end{array}
\right\}
\qquad \text{(K-48)}
$$

Both $\overset{*}{H}_{ij}$ and $\overset{*}{H}_{ji}$ are 6-by-3 matrices. The elements of $\overset{*}{H}_{ij}$ are shown in Eq. (K-44). $\overset{*}{H}_{ji}$ may be obtained from $\overset{*}{H}_{ij}$ by a simple interchange of all subscripts i and j.

Matrices $\overset{*}{J}_{ij}$ and $\overset{*}{K}_{ij}$ relate $\delta \underline{v}_i$ to $\delta \underline{r}_i$ and $\delta \underline{r}_j$.

$$\delta \underline{v}_i = \overset{*}{L}_i \, (\overset{*}{H}_{ij} \;\; \delta \underline{r}_i + \overset{*}{H}_{ji} \;\; \delta \underline{r}_j) \tag{K-45}$$

$$= \overset{*}{J}_{ij} \;\; \delta \underline{r}_i + \overset{*}{K}_{ij} \;\; \delta \underline{r}_j \tag{K-46}$$

The matrix products indicated by (K-45) and (K-46) have been obtained and are recorded as Eqs. (K-47) and (K-48).

The factor X appears in the denominator of each of the in-plane elements of all four matrices, $\overset{*}{H}_{ij}$, $\overset{*}{H}_{ji}$, $\overset{*}{J}_{ij}$, and $\overset{*}{K}_{ij}$. The factor $(\det)_z$ appears in the denominator of all out-of-plane elements. Also, there are elements containing the term $3\, E_M / \sin E_M$. Therefore, the matrices are not applicable when X or $(\det)_z$ or $\sin E_M$ is equal to zero.

The elements of $\overset{*}{K}_{ij}$ in Eq. (K-48) may be compared with those of $\overset{*}{N}_{ji}$ in Eq. (K-40). Let k_{rs} be the element in the r-th row and the s-th column of $\overset{*}{K}_{ij}$.

$$\overset{*}{K}_{ij} = \left\{ \begin{matrix} k_{11} & k_{12} & 0 \\ k_{21} & k_{22} & 0 \\ 0 & 0 & k_{33} \end{matrix} \right\} \tag{K-49}$$

Matrix $\overset{*}{N}_{ji}$ can be expressed in terms of the elements of $\overset{*}{K}_{ij}$.

$$\overset{*}{N}_{ji} = \left\{ \begin{array}{c|c} \dfrac{4 X \sin E_M}{n^2} \begin{matrix} -k_{22} & k_{12} \\ k_{21} & -k_{11} \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline \begin{matrix} 0 & 0 \end{matrix} & \dfrac{1}{k_{33}} \end{array} \right\} \tag{K-50}$$

Equation (F-34) indicates that $\overset{*}{K}_{ij}$ is the inverse of $\overset{*}{N}_{ji}$. Therefore,

$$\overset{*}{K}_{ij} \; \overset{*}{N}_{ji} = \overset{*}{I}_3 \tag{K-51}$$

The off-diagonal elements of $\overset{*}{K}_{ij} \; \overset{*}{N}_{ji}$ are easily verified as being zero from (K-49) and (K-50). It is also obvious that the element in the third row and the third column is unity. The equation for either of the other two diagonal elements yields a simple relationship between the k's and the factor X.

$$\frac{4 X \sin E_M}{n^2} \left(-k_{11} k_{22} + k_{12} k_{21}\right) = 1 \tag{K-52}$$

$$k_{11} k_{22} - k_{12} k_{21} = - \frac{n^2}{4 X \sin E_M} \tag{K-53}$$

The combination $(k_{11} k_{22} - k_{12} k_{21})$ is the determinant of the 2-by-2 sub-matrix of $\overset{*}{K}_{ij}$ which relates to motion in the plane of the reference trajectory. The determinant of the sub-matrix is equal to

$$- \frac{n^2}{4 X \sin E_M} .$$

Then the determinant of the 2-by-2 sub-matrix of $\overset{*}{N}_{ji}$ is

$$- \frac{4 X \sin E_M}{n^2} .$$

The quantity $- 4 X \sin E_M$ has been encountered once before, in Eq. (K-16), where it was indicated as the determinant of the 4-by-4 matrix of Eq. (K-13).

K.10. Checks of the Matrix Elements

Some of the equations developed in Appendix F for the n-body problem may be used as a check of the validity of the matrix formulations of Sections K.8 and K.9. In particular, Eqs. (F-75) through (F-79) may be checked by inspection.

A simple cross-check of $\overset{*}{N}_{ji}$ and $\overset{*}{K}_{ij}$ was made in the last section. The author has verified Eq. (K-53) by actually performing the indicated multiplication of matrix elements. Equation (F-33) has been used to check the elements of $\overset{*}{M}_{ji}$, $\overset{*}{K}_{ij}$, and $\overset{*}{J}_{ij}$.

Additional checks are obtainable from the matrix differential equations of Section F.5. These include Eqs. (F-51) through (F-54), (F-57) and (F-58), (F-63) and (F-64), and (F-69) through (F-72). The matrices $\overset{*}{G}$ and $\overset{*}{W}$ are needed for these checks.

From Eqs. (E-19), (B-81), and (B-82),

$$\overset{*}{G} = \frac{n^2}{(1 - e\cos E)^3} \left\{ \frac{3}{1 - e^2\cos^2 E} \begin{pmatrix} 1 - e^2 & (1 - e^2)^{1/2} e \sin E & 0 \\ (1 - e^2)^{1/2} e \sin E & e^2 \sin^2 E & 0 \\ 0 & 0 & 0 \end{pmatrix} - \overset{*}{I}_3 \right\}$$

(K-54)

The angular velocity of the pqz coordinate system is $\dot{g}\underline{u}_z$. From Eqs. (B-80) and (F-44), the W matrix is given by

$$\overset{*}{W} = \frac{n(1 - e^2)^{1/2}}{(1 + e\cos E)(1 - e\cos E)^2} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{(K-55)}$$

The differential equation checks have not actually been carried out analytically, although spot checks have been made for some of the elements in Eqs. (F-63) and (F-64). In general, the equations serve as a "back-up" in case any element of any matrix is open to question.

APPENDIX L
FIXED-TIME-OF-ARRIVAL GUIDANCE

L.1 Summary

When the destination point is fixed in space and time, the required velocity correction may be expressed in terms of the predicted position variation at the destination by means of the simple matrix equation

$$\underline{c}_F = - \overset{*}{K}_{CD} \, \delta \underline{r}_D^{-} \tag{L-1}$$

where $\underline{c}_F$ is the velocity correction vector for fixed-time-of-arrival guidance. $\delta \underline{r}_D^{-}$ is the position variation vector at the destination which would exist if no correction were applied. $\overset{*}{K}_{CD}$ is a 3-by-3 matrix which can be evaluated numerically for the many-body problem and can be determined analytically for the two-body problem.

L.2 The Velocity Correction

The basic assumption in the guidance theory to be developed is that all variations from the known reference trajectory are small. This assumption holds both before and after the application of a velocity correction. Consequently, the correction itself must be a small quantity.

The velocity correction is assumed to be the result of a thrust impulse. At the time of the correction, the thrust impulse causes an impulse in vehicle acceleration, which in turn produces a step change in vehicle velocity. The correction causes no instantaneous change in vehicle position.

The subscript C appended to a time-varying quantity signifies the value of the quantity corresponding to the time of application of the correction. The superscripts - and + are used, respectively, to indicate conditions existing before and after the correction.

The position and velocity variations at the instant after the correction $\underline{c}$ are related to the variations immediately before the correction as follows:

$$\delta \underline{r}_C^{+} = \delta \underline{r}_C^{-} \tag{L-2}$$

$$\delta \underline{v}_C^{+} = \delta \underline{v}_C^{-} + \underline{c} \tag{L-3}$$

These two relations may be combined into a single equation by use of the six-dimensional vector $\delta \underline{x}$.

$$\delta \underline{x}_C^{+} = \begin{Bmatrix} \delta \underline{r}_C^{+} \\ \\ \delta \underline{v}_C^{+} \end{Bmatrix} = \delta \underline{x}_C^{-} + \begin{Bmatrix} \underline{0}_3 \\ \\ \underline{c} \end{Bmatrix} \tag{L-4}$$

From Eq. (L-3) the velocity correction is given by

$$\underline{c} = \delta \underline{v}_C^{+} - \delta \underline{v}_C^{-} \tag{L-5}$$

The six quantities constituting $\delta \underline{x}_C$ completely define the variant path of the vehicle in the gravitational field. From Eq. (L-4) it is apparent that only three of the six can be altered by the correction $\underline{c}$; hence, only three mathematical conditions can be satisfied by the correction. Many different guidance schemes may be formulated by the simple expedient of varying the conditions to be satisfied by $\underline{c}$.

L. 3 The Velocity Correction for FTA Guidance

For some types of missions, the goal is to have the vehicle arrive at a fixed point (the destination) in heliocentric space at a fixed time. This type is known as a fixed-time-of-arrival (FTA) mission.

The three mathematical conditions to be met in an FTA mission are obviously those involved in reducing to zero the three components of position variation at the destination. In mathematical language, it is desired that

$$\delta \underline{r}_D^+ = \underline{0}_3 \tag{L-6}$$

where the subscript D refers to conditions at the time of arrival at the destination.

The problem now is to determine $\underline{c}_F$ such that Eq. (L-6) is satisfied. Equation (F-29) is used to get expressions for $\delta \underline{v}_C^-$ and $\delta \underline{v}_C^+$, from which $\underline{c}_F$ may be obtained by use of (L-5).

$$\delta \underline{v}_C^- = \overset{*}{J}_{CD}\, \delta \underline{r}_C^- + \overset{*}{K}_{CD}\, \delta \underline{r}_D^- \tag{L-7}$$

$$\delta \underline{v}_C^+ = \overset{*}{J}_{CD}\, \delta \underline{r}_C^+ + \overset{*}{K}_{CD}\, \delta \underline{r}_D^+ \tag{L-8}$$

$$= \overset{*}{J}_{CD}\, \delta \underline{r}_C^- \tag{L-9}$$

$$\underline{c}_F = \delta \underline{v}_C^+ - \delta \underline{v}_C^- = - \overset{*}{K}_{CD}\, \delta \underline{r}_D^- \tag{L-10}$$

It is interesting to note that, although six quantities are needed to specify completely the vehicle's variant path, only three quantities are required to determine the velocity correction vector in FTA guidance. This fact can effect an appreciable saving in computation.

A simple logical argument can be made for the validity of Eq. (L-10) without recourse to mathematics. Since the objective of the guidance system is to reduce $\delta \underline{r}_D$ to zero, it is obvious that $\underline{c}_F$ must be zero if $\delta \underline{r}_D^-$ is zero, and $\underline{c}_F$ must be non-zero if $\delta \underline{r}_D^-$ is non-zero. Therefore, the correction depends on $\delta \underline{r}_D^-$ and is not affected by any characteristics of the variant path that are independent of $\delta \underline{r}_D^-$.

The velocity correction does not, and indeed it cannot, cause the vehicle to return instantaneously to the reference trajectory, inasmuch as such a procedure would require that six, rather than three, conditions be met (i.e., $\delta \underline{r}_C^+ = 0$, $\delta \underline{v}_C^+ = 0$). What the correction does accomplish is to set the vehicle on a new variant path which intersects the original variant path at $t = t_C$ and intersects the reference path at $t = t_D$. This concept is illustrated in Fig. L.1.

L.4 Velocity Variation at the Destination

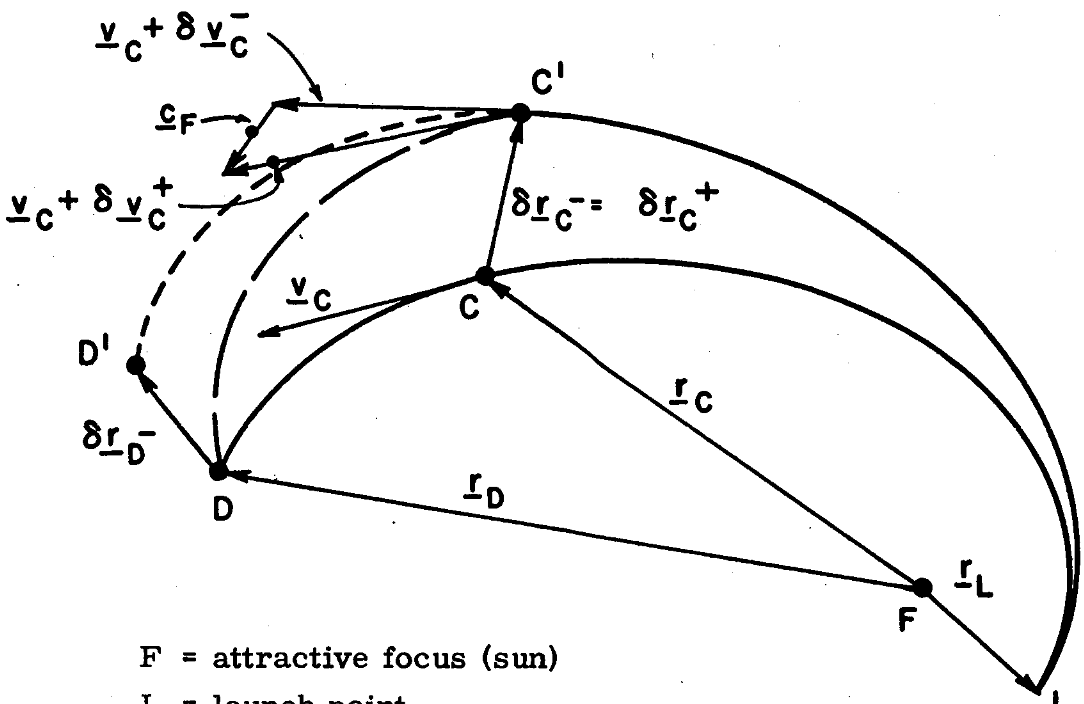
The impulsive thrust correction which nullifies the position variation at the destination does not have the same effect on the velocity variation. From Eq. (F-29), after the correction is applied, the residual velocity variation at the destination is

$$\delta \underline{v}_D^+ = \overset{*}{J}_{DC}\, \delta \underline{r}_D^+ + \overset{*}{K}_{DC}\, \delta \underline{r}_C^+ \tag{L-11}$$

Equations (L-2) and (L-6) are substituted into (L-11).

$$\delta \underline{v}_D^+ = \overset{*}{K}_{DC}\, \delta \underline{r}_C^- \tag{L-12}$$

Thus, for a path deviation vector composed of $\delta \underline{r}_C^-$ and $\delta \underline{r}_D^-$, $\delta \underline{v}_D^+$ depends only on that part of the path deviation vector contained in $\delta \underline{r}_C^-$, while $\underline{c}_F$ depends only on that part contained in $\delta \underline{r}_D^-$.



F = attractive focus (sun)

L = launch point

C = point on reference path corresponding to time of correction t_C

C' = point on actual path corresponding to time of correction t_C

D = destination point

D' = predicted position of vehicle at nominal time of arrival at destination if no correction is applied

LCD = reference path

LC' = actual path from launch to time of correction

C'D' = predicted actual path if no correction is applied

C'D = corrected path

$\delta\underline{r}_C^- = \delta\underline{r}_C^+$ = position variation at time of correction

$\delta\underline{r}_D^-$ = predicted position variation at target if no correction is applied

$\underline{c}_F$ = velocity correction vector

$$= (\underline{v}_C + \delta\underline{v}_C^+) - (\underline{v}_C + \delta\underline{v}_C^-) = \delta\underline{v}_C^+ - \delta\underline{v}_C^-$$

Figure L.1 Fixed-Time-of-Arrival Guidance

Since the most practical method of expressing the characteristics of the original variant path is in terms of the components of $\delta \underline{x}_D^-$, it is desirable to express $\delta \underline{v}_D^+$ in terms of $\delta \underline{r}_D^-$ and $\delta \underline{v}_D^-$, rather than $\delta \underline{r}_C^-$. Such an expression is readily obtained from the difference $(\delta \underline{v}_D^+ - \delta \underline{v}_D^-)$.

$$\delta \underline{v}_D^+ - \delta \underline{v}_D^- = \overset{*}{J}_{DC} (\delta \underline{r}_D^+ - \delta \underline{r}_D^-) + \overset{*}{K}_{DC} (\delta \underline{r}_C^+ - \delta \underline{r}_C^-)$$

$$= - \overset{*}{J}_{DC} \, \delta \underline{r}_D^- \tag{L-13}$$

$$\delta \underline{v}_D^+ = - \overset{*}{J}_{DC} \, \delta \underline{r}_D^- + \delta \underline{v}_D^- = \{ -\overset{*}{J}_{DC} \quad \overset{*}{I}_3 \} \, \delta \underline{x}_D^- \tag{L-14}$$

Equations (L-6) and (L-14) may be combined into a single equation relating $\delta \underline{x}_D^+$ to $\delta \underline{x}_D^-$.

$$\delta \underline{x}_D^+ = \begin{Bmatrix} \overset{*}{O}_3 & \overset{*}{O}_3 \\ -\overset{*}{J}_{DC} & \overset{*}{I}_3 \end{Bmatrix} \delta \underline{x}_D^- \tag{L-15}$$

L. 5 Change in the Orbital Elements

When the reference trajectory is an ellipse, it may be of interest to determine the change in the orbital elements caused by the correction $\underline{c}_F$. From Eqs. (F-1) and (F-2), the original path deviation vector $\delta \underline{e}^-$ may be written as

$$\delta \underline{e}^- = \{ \overset{*}{R}_D \quad \overset{*}{V}_D \} \, \delta \underline{x}_D^- \tag{L-16}$$

$$= \{ \overset{*}{H}_{CD} \quad \overset{*}{H}_{DC} \} \begin{Bmatrix} \delta \underline{r}_C^- \\ \delta \underline{r}_D^- \end{Bmatrix} \tag{L-17}$$

After the corrective thrust is applied, the path deviation vector becomes

$$\delta \underline{e}^{+} = \{ \overset{*}{H}_{CD} \qquad \overset{*}{H}_{DC} \} \begin{Bmatrix} \delta \underline{r}_{C}^{+} \\ \delta \underline{r}_{D}^{+} \end{Bmatrix}$$

$$= \overset{*}{H}_{CD} \; \delta \underline{r}_{C}^{-} \tag{L-18}$$

The change in the variations of the orbital elements is

$$\delta \underline{e}^{+} - \delta \underline{e}^{-} = - \overset{*}{H}_{DC} \, \delta \underline{r}_{D}^{-} \tag{L-19}$$

Like $\underline{c}_F$, $(\delta \underline{e}^{+} - \delta \underline{e}^{-})$ depends on $\delta \underline{r}_D^{-}$ and no other parameters of the original variant trajectory. The close relationship between $\underline{c}_F$ and $(\delta \underline{e}^{+} - \delta \underline{e}^{-})$ is clearly shown by means of Eq. (F-17).

$$\underline{c}_F = \delta \underline{v}_C^{+} - \delta \underline{v}_C^{-} = \overset{*}{L}_C \, (\delta \underline{e}^{+} - \delta \underline{e}^{-}) \tag{L-20}$$

Equations (L-10) and (L-19) can be combined to obtain a relationship that is the inverse of (L-20). Equation (F-34) is used to simplify the result.

$$\delta \underline{e}^{+} - \delta \underline{e}^{-} = \overset{*}{H}_{DC} \, \overset{*}{K}_{CD}{}^{-1} \, \underline{c}_F = \overset{*}{H}_{DC} \, \overset{*}{N}_{DC} \, \underline{c}_F \tag{L-21}$$

L.6 Method of Numerical Evaluation

Once $\delta \underline{r}_D^-$ has been determined, the correction $\underline{c}_F$ corresponding to any given t_C can be computed as soon as the elements of $\overset{*}{K}_{CD}$ have been evaluated.

For the many-body problem the elements of $\overset{*}{K}_{CD}$ are computed by numerical integration, as shown in Section F.6. In accordance with the suggestion made in that section, the equations to be integrated are simplified by using a non-rotating coordinate system, and the round-off error is reduced by choosing the z-axis to be perpendicular to the plane of the basic motion (i.e., the motion that would exist in the absence of disturbing forces).

From (F-53) and (F-54), the matrix differential equations are

$$\frac{\partial \overset{*}{N}_{CD}}{\partial t_C} = \overset{*}{T}_{CD} \tag{L-22}$$

$$\frac{\partial \overset{*}{T}_{CD}}{\partial t_C} = \overset{*}{G}_C \overset{*}{N}_{CD} \tag{L-23}$$

These equations are integrated in the negative time direction, starting from $t_C = t_D$. For a given reference trajectory, t_D is a fixed quantity, and the selected time for applying the correction lies in the range t_I to t_D, where t_I is the time of injection. The initial conditions are

$$\overset{*}{N}_{DD} = \overset{*}{O}_3 \qquad \overset{*}{T}_{DD} = \overset{*}{I}_3 \tag{L-24}$$

For the non-rotating coordinate system, the elements of $\overset{*}{G}_C$ are known as a function of t_C from Eq. (E-11). The matrix combination inside the braces on the right side of (E-11), when evaluated at $t = t_C$, constitutes $\overset{*}{G}_C$.

The two matrix equations (L-22) and (L-23) consist of eighteen coupled first-order differential equations in the eighteen variables composed of the elements of $\overset{*}{N}_{CD}$ and $\overset{*}{T}_{CD}$. The numerical integration yields these elements as a function of t_C.

$\overset{*}{K}_{DC}$ and $\overset{*}{K}_{CD}$ are obtained from $\overset{*}{N}_{CD}$ by simple matrix manipulation.

From (F-34),

$$\overset{*}{K}_{DC} = \overset{*}{N}_{CD}{}^{-1} \tag{L-25}$$

From (F-79),

$$\overset{*}{K}_{CD} = - \overset{*}{K}_{DC}{}^{T} = - (\overset{*}{N}_{CD}{}^{T})^{-1} \tag{L-26}$$

If the reference trajectory is an ellipse, there is no need for the numerical integration. An analytic solution for the elements of $\overset{*}{K}_{CD}$, in the flight path coordinate system, may be obtained by the proper substitution of subscripts in Eqs. (K-14), (K-15), and (K-48).

The mechanization of the guidance system does not require a knowledge of $\delta \underline{v}_D{}^+$. However, such knowledge is of value if more than one midcourse correction is to be applied.

The additional information needed to compute $\delta \underline{v}_D{}^+$ includes the components of $\delta \underline{v}_D{}^-$ and the elements of $\overset{*}{J}_{DC}$. $\delta \underline{v}_D{}^-$, like $\delta \underline{r}_D{}^-$, is based on the observations made during the course of the flight. The determination of $\overset{*}{J}_{DC}$ involves a procedure similar to the one described for evaluating $\overset{*}{K}_{CD}$.

For the many-body solution, eighteen additional coupled first-order differential equations are integrated numerically in the negative time direction, starting from $t = t_D$. The eighteen are contained in two matrix differential equations derived from (F-51) and (F-52).

$$\frac{\partial \overset{*}{M}_{CD}}{\partial t_C} = \overset{*}{S}_{CD} \tag{L-27}$$

$$\frac{\partial \overset{*}{S}_{CD}}{\partial t_C} = \overset{*}{G}_C \overset{*}{M}_{CD} \tag{L-28}$$

The initial conditions are

$$\overset{*}{M}_{DD} = \overset{*}{I}_3 \qquad \overset{*}{S}_{DD} = \overset{*}{O}_3 \tag{L-29}$$

The solution contains the elements of $\overset{*}{M}_{CD}$ and $\overset{*}{S}_{CD}$ as a function of t_C. $\overset{*}{J}_{DC}$ is obtained from $\overset{*}{K}_{CD}$ and $\overset{*}{M}_{CD}$ by the use of Eq. (F-33).

$$\overset{*}{J}_{DC} = - \overset{*}{K}_{DC} \overset{*}{M}_{CD} = \overset{*}{K}_{CD}{}^T \overset{*}{M}_{CD} \tag{L-30}$$

A check on the compuations is afforded by the fact that $\overset{*}{J}_{DC}$ is a symmetric matrix.

For an elliptical reference trajectory, the analytic form of $\overset{*}{J}_{DC}$, in the flight path coordinate system, comes directly from Eqs. (K-14), (K-15), and (K-47).

Numerical evaluation of the elements of $\delta \underline{e}$, either before or after the correction, is not necessary for the mechanization of the guidance system. If for some reason the numerical values are desired, the matrices $\overset{*}{R}_D$ and $\overset{*}{V}_D$ are required to determine $\delta \underline{e}^-$ from Eq. (L-17), and $\overset{*}{H}_{CD}$ is needed to determine $\delta \underline{e}^+$ from Eq. (L-18). Analytical expressions for $\overset{*}{R}_D$, $\overset{*}{V}_D$, and $\overset{*}{H}_{CD}$ may be obtained from Eqs. (K-34), (K-35), and (K-44), respectively.

APPENDIX M

VARIABLE-TIME-OF-ARRIVAL GUIDANCE

M.1 Summary

When the nature of the space mission is such that the time of arrival at the destination need not be rigidly constrained, the velocity correction may be expressed in terms of only two components of the predicted position variation at the nominal time of arrival. The correction can be computed in such a way that, for the particular time of correction selected, the magnitude of the correction is minimized. This method of computation is known as variable-time-of-arrival (VTA) guidance.

Equations are developed for the velocity correction in VTA guidance and also for the change in the time of arrival.

M.2 Design Philosophy of VTA Guidance

The concept of VTA guidance is clarified by the introduction of two new vectors, the relative velocity vector $\underline{v}_R$ and the miss distance vector $\delta \underline{\rho}$.

$\underline{v}_R$ is the relative velocity of the space vehicle, on its reference trajectory, with respect to the destination planet at the nominal time of arrival at the destination. In mathematical terms,

$$\underline{v}_R = \underline{v}_S - \underline{v}_P \tag{M-1}$$

where $\underline{v}_S$ is the velocity of the space vehicle on its reference trajectory at the nominal time of arrival and $\underline{v}_P$ is the velocity of the destination planet at that time. Fig. M.1 gives a schematic representation of $\underline{v}_R$.

$\delta \underline{\rho}$ is defined as the component of $\delta \underline{r}_D$, the position variation vector at the destination, that is perpendicular to $\underline{v}_R$. It represents the minimum distance between vehicle and destination point.

The objective of VTA guidance is to reduce $\delta \underline{\rho}$ to zero. Since $\delta \underline{\rho}$ lies in the plane perpendicular to $\underline{v}_R$, accomplishing this objective accounts for only two of the three conditions that can be satisfied by the velocity correction. A third condition must be specified before the correction can be determined uniquely.

Although there are several practical possibilities for the third condition, as indicated in References (9) and (10), the only one considered in this analysis is the minimization of the magnitude of the midcourse velocity correction.

M.3 Basic Guidance Equations for VTA Guidance

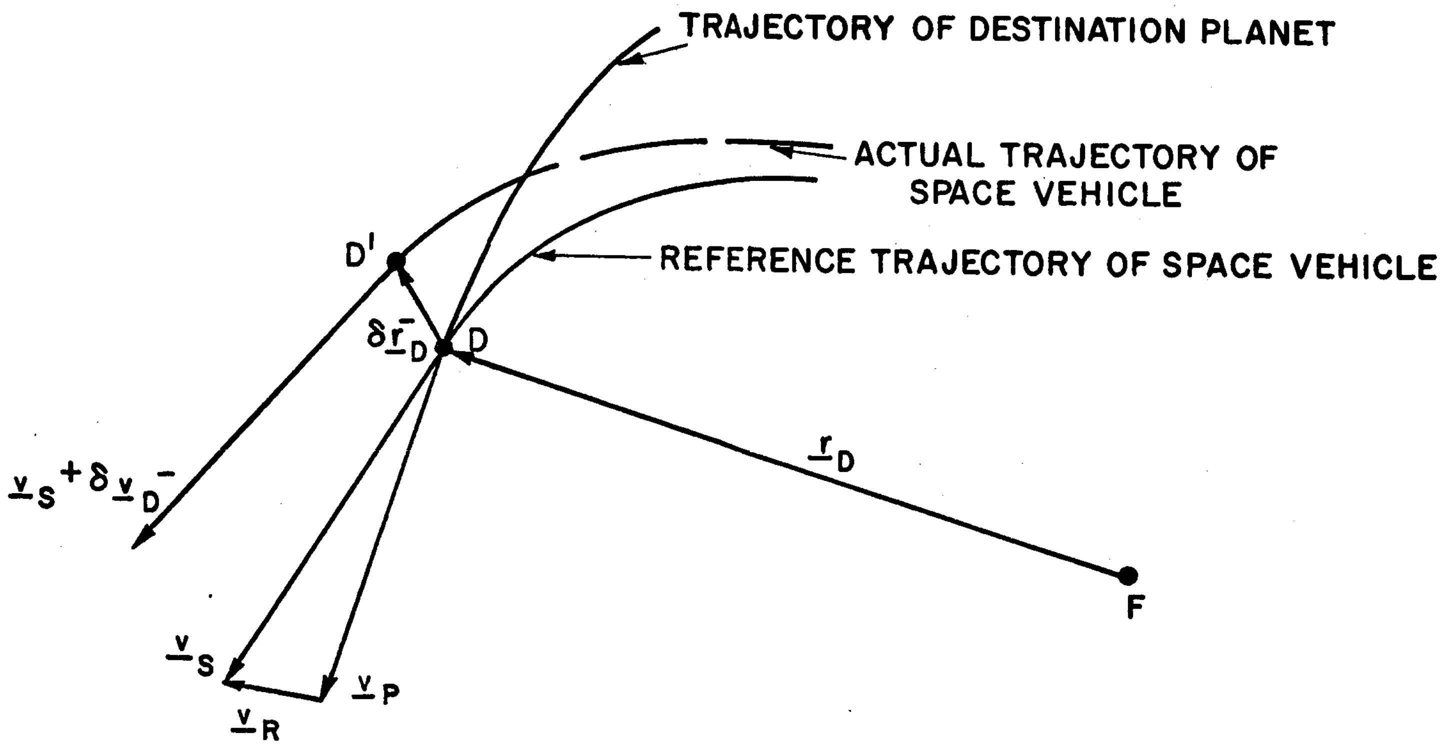
The change in the time of arrival due to VTA guidance is designated Δt_D. The variation Δt_D, unlike the variational quantities previously discussed, is deliberately inserted into the system; the use of the symbol Δ rather than δ is intended to emphasize this distinction. The actual time of arrival at the destination in VTA guidance is $t_D + \Delta t_D$.

Prior to the application of the correction, the predicted velocity of the vehicle relative to the destination planet at $t = t_D$ is $\underline{v}_R + \delta \underline{v}_D^-$. After the correction is applied, the relative velocity at $t = t_D$ is $\underline{v}_R + \delta \underline{v}_D^+$.

With both Δt_D and $\delta \underline{v}_D^+$ recognized as small variational quantities, linear theory gives the following relationship for $\delta \underline{r}_D^+$ in VTA guidance:

$$\delta \underline{r}_D^+ = -(\underline{v}_R + \delta \underline{v}_D^+) \Delta t_D$$

$$= -\underline{v}_R \Delta t_D \tag{M-2}$$



F = attractive focus (sun)

D = destination point on reference trajectory

D' = predicted position of vehicle at nominal time of arrival at destination ($t = t_D$)

$\delta \underline{r}_D^-$ = predicted position variation at $t = t_D$

$\delta \underline{v}_D^-$ = predicted velocity variation at $t = t_D$

$\underline{v}_S$ = velocity of space vehicle on reference trajectory at $t = t_D$

$\underline{v}_P$ = velocity of destination planet at $t = t_D$

$\underline{v}_R$ = relative velocity of space vehicle with respect to destination planet at $t = t_D$

$= \underline{v}_S - \underline{v}_P$

Figure M.1 Relative Velocity Vector

In Fig. M. 2, the VTA correction moves the predicted vehicle position at $t = t_D$ from D' to H. The distance of H from D, the position of the destination planet at $t = t_D$, is the magnitude of $\delta \underline{r}_D^+$. Note that $\delta \underline{r}_D^+$ is not, in general, equal in magnitude to the component of $\delta \underline{r}_D^-$ in the $\underline{v}_R$ direction.

Figure M. 2 illustrates the basic difference between the FTA and VTA systems. The correction in FTA guidance is made such that the vehicle passes through the specific point D at $t = t_D$, while the correction in the general concept of VTA guidance requires only that at $t = t_D$ the vehicle be situated on the line through D parallel to $\underline{v}_R$.

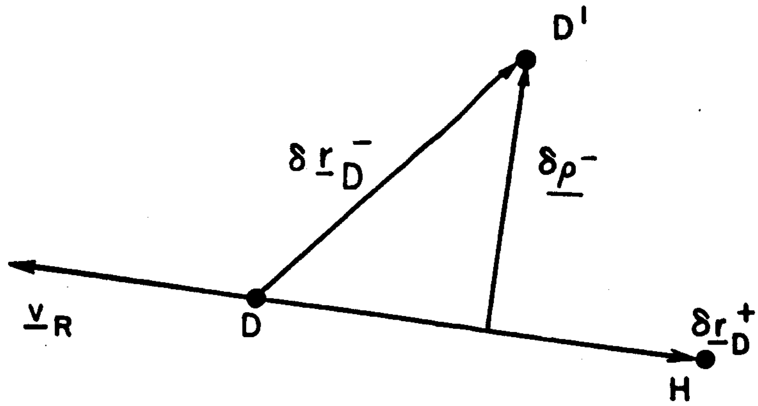
Let $\underline{c}_V$ denote the velocity correction in VTA guidance.

$$\begin{aligned} \underline{c}_V &= \delta \underline{v}_C^+ - \delta \underline{v}_C^- \\ &= (\overset{*}{J}_{CD}\, \delta \underline{r}_C^+ + \overset{*}{K}_{CD}\, \delta \underline{r}_D^+) - (\overset{*}{J}_{CD}\, \delta \underline{r}_C^- + \overset{*}{K}_{CD} \cdot \delta \underline{r}_D^-) \\ &= \overset{*}{K}_{CD}\, (\delta \underline{r}_D^+ - \delta \underline{r}_D^-) \qquad \text{(M-3)} \\ &= \underline{c}_F - \underline{w}\, \Delta t_D \qquad \text{(M-4)} \end{aligned}$$

where

$$\underline{w} = \overset{*}{K}_{CD}\, \underline{v}_R \qquad \text{(M-5)}$$

Equations (M-3), (M-4), and (M-5) are the basic equations of VTA guidance, independent of the third condition to be satisfied by the correction vector $\underline{c}_V$. Specifying a third condition is analogous to specifying Δt_D; when this is done, $\underline{c}_V$ is determined uniquely.



D = nominal destination point

D' = predicted position of vehicle at $t = t_D$ if no correction is applied

H = predicted position of vehicle at $t = t_D$ if VTA correction is applied at $t = t_C$

$\delta \underline{r}_D^-$ = predicted position variation vector at $t = t_D$ if no correction is applied

$\delta \underline{\rho}^-$ = miss distance vector

= component of $\delta \underline{r}_D^-$ perpendicular to $\underline{v}_R$

$\underline{v}_R$ = nominal velocity vector of vehicle relative to destination point at $t = t_D$

$\delta \underline{r}_D^+$ = predicted position variation vector at $t = t_D$ if VTA correction is applied at $t = t_C$

Figure M.2 Miss Distance Vector and VTA Guidance

M.4 Variation in Time of Arrival

The variation in time of arrival is to be determined such that it satisfies the condition that the magnitude of $\underline{c}_V$ be a minimum.

From Eq. (M-4),

$$c_V^2 = \underline{c}_V^T \underline{c}_V = \underline{c}_F^T \underline{c}_F - 2 \underline{w}^T \underline{c}_F \Delta t_D + \underline{w}^T \underline{w} (\Delta t_D)^2 \qquad \text{(M-6)}$$

The partial derivative of c_V^2 with respect to Δt_D is equated to zero. The vectors $\underline{c}_F$ and $\underline{w}$ are both independent of Δt_D.

$$\frac{\partial (c_V^2)}{\partial (\Delta t_D)} = 0 = - 2 \underline{w}^T \underline{c}_F + 2 \underline{w}^T \underline{w} \Delta t_D \qquad \text{(M-7)}$$

The solution of this equation for Δt_D is

$$\Delta t_D = \frac{\underline{w}^T \underline{c}_F}{\underline{w}^T \underline{w}} \qquad \text{(M-8)}$$

M.5 Velocity Correction in VTA Guidance

Equation (M-8) may be substituted into Equation (M-4).

$$\underline{c}_V = \underline{c}_F - \frac{\underline{w}\, \underline{w}^T}{\underline{w}^T \underline{w}} \underline{c}_F = \left(\overset{*}{I}_3 - \frac{\underline{w}\, \underline{w}^T}{\underline{w}^T \underline{w}} \right) \underline{c}_F \qquad \text{(M-9)}$$

$$= - \left(\overset{*}{I}_3 - \frac{\underline{w}\, \underline{w}^T}{\underline{w}^T \underline{w}} \right) \overset{*}{K}_{CD} \, \delta \underline{r}_D^- \qquad \text{(M-10)}$$

Equation (M-9) shows the mathematical relationship between VTA and FTA velocity corrections. It was developed in this form by Battin [4].

An interesting result is obtained from the scalar product of $\underline{c}_V$ with $\underline{w}$.

$$\underline{w} \cdot \underline{c}_V = \underline{w}^T \underline{c}_V = \left(\underline{w}^T - \frac{\underline{w}^T \underline{w}}{\underline{w}^T \underline{w}} \underline{w}^T \right) \underline{c}_F = 0 \qquad \text{(M-11)}$$

Since neither the vector $\underline{c}_V$ nor the vector $\underline{w}$ is in general a zero vector, it is apparent that $\underline{c}_V$ is always perpendicular to $\underline{w}$. Thus, $\underline{c}_V$ is constrained to lie in the plane perpendicular to $\underline{w}$. Noton[9] refers to the direction of $\underline{w}$ as the "noncritical direction" and the plane normal to $\underline{w}$ as the "critical plane".

The vector $\underline{w}$ depends on $\overset{*}{K}_{CD}$, which is a function of both t_C and t_D. For a specified reference trajectory, t_D is fixed, but t_C can vary. Consequently, the noncritical direction and the orientation of the critical plane both depend on the time at which the correction is to be made.

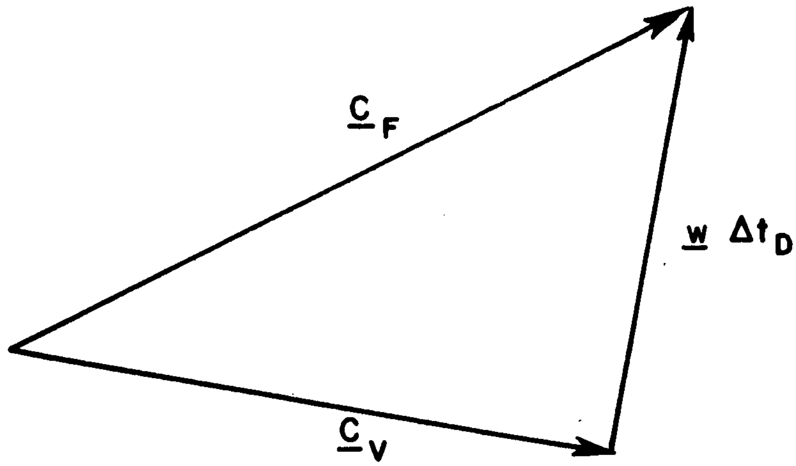
From (M-4), $\underline{c}_F$ is the vector sum of $\underline{c}_V$ and $\underline{w}\,\Delta t_D$. Since Δt_D is a scalar, the vector $\underline{w}\,\Delta t_D$ is parallel to $\underline{w}$. The other term in the vector sum, namely $\underline{c}_V$, is perpendicular to $\underline{w}$. Thus, the vector triangle, shown in Fig. M.3, is a right triangle whose hypotenuse is $\underline{c}_F$, and $\underline{c}_V$ is simply the component of $\underline{c}_F$ in the critical plane.

M.6 Position Variation and Velocity Variation at the Destination

The position variation $\delta \underline{r}_D^{+}$ can be expressed as a function of $\delta \underline{r}_D^{-}$ by combining Eqs. (M-2), (M-8), and (L-1).

$$\begin{aligned} \delta \underline{r}_D^{+} &= - \underline{v}_R \, \Delta t_D \\ &= - \frac{\underline{v}_R \underline{w}^T}{\underline{w}^T \underline{w}} \underline{c}_F \end{aligned} \qquad \text{(M-12)}$$

$$= \frac{\underline{v}_R \underline{w}^T}{\underline{w}^T \underline{w}} \overset{*}{K}_{CD} \, \delta \underline{r}_D^{-} \qquad \text{(M-13)}$$



$\underline{c}_F$ = FTA velocity correction vector

$\underline{c}_V$ = VTA velocity correction vector

$\underline{w} = \overset{*}{K}_{CD}\, \underline{v}_R$ = vector in noncritical direction

$\overset{*}{K}_{CD}$ = 3-by-3 matrix depending on t_C and t_D

$\underline{v}_R$ = relative velocity vector

Δt_D = change in time of arrival at destination

Figure M.3 Vector Relation between Velocity Corrections in FTA and VTA Guidance

The velocity deviation $\delta \underline{v}_D^{\,+}$ is

$$\delta \underline{v}_D^{\,+} = \overset{*}{J}_{DC} \, \delta \underline{r}_D^{\,+} + \overset{*}{K}_{DC} \, \delta \underline{r}_C^{\,+} \tag{M-14}$$

From Eqs. (L-2), (L-12), and (L-14),

$$\overset{*}{K}_{DC} \, \delta \underline{r}_C^{\,+} = \overset{*}{K}_{DC} \, \delta \underline{r}_C^{\,-} = \left\{ -\overset{*}{J}_{DC} \qquad \overset{*}{I}_3 \right\} \delta \underline{x}_D^{\,-} \tag{M-15}$$

The final expression for $\delta \underline{v}_D^{\,+}$ is the result of combining (M-13), (M-14), and (M-15).

$$\delta \underline{v}_D^{\,+} = -\overset{*}{J}_{DC} \left(\overset{*}{I}_3 - \frac{\underline{v}_R \, \underline{w}^T}{\underline{w}^T \, \underline{w}} \overset{*}{K}_{CD} \right) \delta \underline{r}_D^{\,-} + \delta \underline{v}_D^{\,-} \tag{M-16}$$

A composite equation can now be written in which $\delta \underline{x}_D^{\,+}$ is expressed in terms of $\delta \underline{x}_D^{\,-}$.

$$\delta \underline{x}_D^{\,+} = \left\{ \begin{array}{cc} \dfrac{\underline{v}_R \, \underline{w}^T}{\underline{w}^T \, \underline{w}} \overset{*}{K}_{CD} & \overset{*}{O}_3 \\ \\ -\overset{*}{J}_{DC} \left(\overset{*}{I}_3 - \dfrac{\underline{v}_R \, \underline{w}^T}{\underline{w}^T \, \underline{w}} \overset{*}{K}_{CD} \right) & \overset{*}{I}_3 \end{array} \right\} \delta \underline{x}_D^{\,-} \tag{M-17}$$

This equation can be compared with Eq. (L-15), the corresponding expression for FTA guidance.

M.7 Change in the Orbital Elements

The six-component vector $\delta\underline{e}^-$, expressing the variations in the orbital elements before application of corrective thrust, is obviously unaffected by the type of correction that is contemplated. It can be expressed in terms of $\delta\underline{x}_D^-$ or in terms of $\delta\underline{r}_C^-$ and $\delta\underline{r}_D^-$, as indicated in Eqs. (L-16) and (L-17).

After the correction, the new vector $\delta\underline{e}^+$ for VTA must differ from the $\delta\underline{e}^+$ for FTA, since different corrections are applied. For VTA,

$$\delta\underline{e}^+ = \{\overset{*}{H}_{CD} \qquad \overset{*}{H}_{DC}\} \begin{Bmatrix} \delta\underline{r}_C^+ \\ \\ \delta\underline{r}_D^+ \end{Bmatrix}$$

$$= \left\{\overset{*}{H}_{CD} \qquad \overset{*}{H}_{DC} \frac{\underline{v}_R \, \underline{w}^T}{\underline{w}^T \, \underline{w}} \overset{*}{K}_{CD}\right\} \begin{Bmatrix} \delta\underline{r}_C^- \\ \\ \delta\underline{r}_D^- \end{Bmatrix} \tag{M-18}$$

The change in $\delta\underline{e}$ due to the VTA correction is obtained by subtracting (L-17) from (M-18).

$$\delta\underline{e}^+ - \delta\underline{e}^- = -\overset{*}{H}_{DC}\left(\overset{*}{I}_3 - \frac{\underline{v}_R \, \underline{w}^T}{\underline{w}^T \, \underline{w}} \overset{*}{K}_{CD}\right) \delta\underline{r}_D^- \tag{M-19}$$

The analogous equations to (L-20) and (L-21) are valid for VTA.

$$\underline{c}_V = \overset{*}{L}_C (\delta \underline{e}^+ - \delta \underline{e}^-) \tag{M-20}$$

$$\delta \underline{e}^+ - \delta \underline{e}^- = \overset{*}{H}_{DC} \overset{*}{K}_{CD}{}^{-1} \underline{c}_V = \overset{*}{H}_{DC} \overset{*}{N}_{DC} \underline{c}_V \tag{M-21}$$

M. 8 Numerical Evaluation

The number of quantities to be evaluated for VTA guidance is obviously greater than the number in FTA guidance. Foremost is the correction $\underline{c}_V$. Second in importance is $\delta \underline{x}_D{}^+$, which now includes a non-zero $\delta \underline{r}_D{}^+$. Third is the change in arrival time, Δt_D. Finally, for elliptical trajectories, there is the capability, though not the necessity, of computing $\delta \underline{e}^-$ and $\delta \underline{e}^+$.

The matrices required for the first three are $\overset{*}{K}_{CD}$ and $\overset{*}{J}_{DC}$, the evaluation of which has been described in Section L. 6. The new quantities involved are the vectors $\underline{v}_R$ and $\underline{w}$. The former is obtained directly from the reference trajectory, and the latter comes from the matrix product of Eq. (M-5). The matrices $\overset{*}{R}_D$, $\overset{*}{V}_D$, and $\overset{*}{H}_{CD}$, needed to evaluate $\delta \underline{e}^-$ and $(\delta \underline{e}^+ - \delta \underline{e}^-)$ are obtained from Appendix K.

It is quite obvious that VTA guidance entails somewhat more computation than FTA guidance. The added steps, however, are simple ones; they consist primarily of multiplications and additions of 3-by-3 matrices; there are no new matrix inversions, and the additional divisions all involve the same scalar quantity, $\underline{w}^T \underline{w}$.

APPENDIX N

OPTIMIZATION OF TIME OF CORRECTION

N.1 Summary

A new rotating coordinate system, called the critical-plane coordinate system, is introduced, in which the VTA velocity correction is expressed as a two-dimensional vector and the miss distance is also expressed as a two-dimensional vector. Then the matrix relating the correction vector to the miss distance vector is reduced to a 2-by-2 matrix. For elliptical reference trajectories, one of the four elements of this matrix is equal to zero.

If the two-dimensional miss distance vector is represented by a magnitude and a phase angle, the magnitude of the correction vector is a linear function of the magnitude of the miss distance vector but varies in a non-linear fashion with the phase angle of the miss distance vector. A technique is developed for determining the time of correction as a function of the phase angle such that the magnitude of the VTA correction is minimized.

N.2 Introduction

Appendix M develops the method of computing the VTA velocity correction corresponding to a given time t_C, but no consideration has yet been given to the means of specifying t_C. Since the magnitude of the correction $\underline{c}_V$ varies with t_C, it is desirable to specify that particular t_C for which the magnitude of $\underline{c}_V$ is minimized. The minimization procedure is facilitated by the introduction of the critical-plane coordinate system.

N.3 Critical-Plane Coordinate System

The axes of the critical-plane coordinate system are designated ξ_C, η_C, and ζ_C. The $\xi_C - \eta_C$ plane is the critical plane corresponding to the given t_C. The ζ_C-axis is in the noncritical direction; i.e., it is parallel to $\underline{w}$. From Eq. (M-5),

$$\underline{w} = \overset{*}{K}_{CD}\,\underline{v}_R \tag{N-1}$$

Since $\overset{*}{K}_{CD}$ varies with t_C, the critical-plane coordinate system is a rotating system. Its origin, like that of the three reference trajectory systems of Appendix A, is at the center of the sun. The ξ_C-axis lies along the line of nodes between the critical plane and the reference trajectory plane.

In the analysis of Appendix M, it may be assumed that one of the three reference trajectory coordinate systems described in Appendix A is used. Let r_1, r_2, r_3 indicate the three axes of the particular system being used. Then Euler angles Ω_C and i_C serve to orient the critical-plane coordinate system with respect to the r_1, r_2, r_3 system. Ω_C is the angle measured in the reference trajectory plane from the positive r_1-axis to the positive ξ_C-axis. i_C is the angle between the positive r_3-axis (i.e., the z-axis) and the positive ζ_C-axis.

The positive ζ_C-axis is in the direction of $\underline{w}$. The positive ξ_C-axis is chosen as that half of the line of nodes for which Ω_C lies between 0° and 180°. (Ω_C is positive in the direction of vehicle motion.) The positive η_C-axis is such that ξ_C, η_C, ζ_C form a right-handed orthogonal triad. It may be noted that i_C, as well as Ω_C, lies in the range 0° to 180°.

N.4 Critical-Plane System Coordinate Axes at Nominal Time of Arrival

The orientation of the critical-plane system coordinate axes depends on $\underline{w}$, which depends on $\overset{*}{K}_{CD}$. For all values of t_C for which the elements of $\overset{*}{K}_{CD}$ can be determined, the axes are defined uniquely. However, if the elements of $\overset{*}{K}_{CD}$ cannot be determined, some other means must be used for specifying the axis directions. Such a problem arises when $t_C = t_D$.

$\overset{*}{K}_{CD}$ is computed from the equation

$$\overset{*}{K}_{CD} = -\left(\overset{*}{N}_{CD}{}^{T}\right)^{-1} \qquad \text{(N-2)}$$

and $\overset{*}{N}_{CD}$ is obtained by integration of eighteen coupled first-order differential equations. At $t_C = t_D$ the matrix $\overset{*}{N}_{DD}$ is the zero matrix; hence it has no finite inverse, and $\overset{*}{K}_{DD}$ cannot be determined.

A physical, rather than purely mathematical, approach can be used effectively to attack this problem. If the vehicle's position at t_D is along the line through the nominal destination point and parallel to $\underline{v}_R$, the objective of the VTA guidance system has been attained, and no further correction is desired. Thus, the non-critical ζ_D-axis is in the direction of $\underline{v}_R$, and the critical plane (i.e., the $\xi_D - \eta_D$ plane) is perpendicular to $\underline{v}_R$.

For the case of elliptical reference trajectories, a mathematical explanation is possible. Let t_C be very close to t_D, so that E_M, which is equal to half the difference between E_D and E_C, is a small angle. For small values of E_M,

$$\sin E_M = E_M \tag{N-3}$$

$$\cos E_M = 1 \tag{N-4}$$

$$\cos E_C = \cos (E_D - 2 E_M) = \cos E_D + 2 E_M \sin E_D \tag{N-5}$$

$$\cos E_P = \cos (E_D - E_M) = \cos E_D + E_M \sin E_D \tag{N-6}$$

When these relations are substituted into the negative transpose of Eq. (K-40), $\overset{*}{N}_{CD}$ for small E_M becomes

$$\overset{*}{N}_{CD} = - \frac{2(1 - e \cos E_D) E_M}{n} \begin{Bmatrix} 1 & - \dfrac{2(1-e^2)^{1/2} E_M}{1 - e^2 \cos^2 E_D} & 0 \\ \dfrac{2(1-e^2)^{1/2} E_M}{1 - e^2 \cos^2 E_D} & 1 & 0 \\ 0 & 0 & 1 \end{Bmatrix} \tag{N-7}$$

In the limit as t_C approaches t_D,

$$\overset{*}{N}_{CD} \longrightarrow -\frac{2(1 - e \cos E_D)\, E_M}{n} \overset{*}{I}_3 \tag{N-8}$$

From Eq. (N-2), when t_C approaches t_D,

$$\overset{*}{K}_{CD} \longrightarrow \frac{n}{2(1 - e \cos E_D)\, E_M} \overset{*}{I}_3 \tag{N-9}$$

When $t_C = t_D$, $E_M = 0$, so that $\overset{*}{K}_{DD}$ is given by

$$\overset{*}{K}_{DD} = \infty \overset{*}{I}_3 \tag{N-10}$$

Substitution of (N-10) into (N-1) indicates that the $\underline{w}$ vector corresponding to $t_C = t_D$ is infinite in magnitude and parallel to $\underline{v}_R$. Therefore, the ζ_D-axis is parallel to $\underline{v}_R$, in agreement with the result obtained by physical reasoning.

N.5 Transformation Relations

The 3-by-3 matrix for transforming from r_1, r_2, r_3 coordinates to ξ, η, ζ coordinates at any specified time will be designated $\overset{*}{X}$.

$$\overset{*}{X} = \begin{Bmatrix} \cos\Omega & \sin\Omega & 0 \\ -\sin\Omega\cos i & \cos\Omega\cos i & \sin i \\ \sin\Omega\sin i & -\cos\Omega\sin i & \cos i \end{Bmatrix} \tag{N-11}$$

$\overset{*}{X}$ is an orthogonal matrix; therefore,

$$\overset{*}{X}^{-1} = \overset{*}{X}^{T} \tag{N-12}$$

Subscript W will be used to indicate that a vector is expressed in terms of its components in the critical-plane coordinate system.

The vector $\underline{w}$ for a given t_C transforms as follows:

$$(\underline{w})_W = \overset{*}{X}_C \, \underline{w} = w \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \tag{N-13}$$

where $\overset{*}{X}_C$ is the transformation matrix evaluated at $t = t_C$. The transformation for $\underline{v}_R$ is

$$(\underline{v}_R)_W = \overset{*}{X}_D \, \underline{v}_R = v_R \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \tag{N-14}$$

$\overset{*}{X}_D$ is evaluated at $t = t_D$. (N-13) may be combined with (N-1).

$$(\underline{w})_W = \overset{*}{X}_C \, \overset{*}{K}_{CD} \, \overset{*}{X}_D{}^{-1} \, (\underline{v}_R)_W \tag{N-15}$$

$$w \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} = v_R \, \overset{*}{X}_C \, \overset{*}{K}_{CD} \, \overset{*}{X}_D{}^T \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \tag{N-16}$$

The matrix product $\overset{*}{X}_C \, \overset{*}{K}_{CD} \, \overset{*}{X}_D{}^T$, itself a 3-by-3 matrix, appears in the equation for $(\underline{c}_V)_W$ which will be derived in the next section. Analytic expressions for the elements of $\overset{*}{X}_C \, \overset{*}{K}_{CD} \, \overset{*}{X}_D{}^T$ can be found in terms of the fixed angles Ω_D and i_D and the time-varying elements of $\overset{*}{K}_{CD}$.

From (N-16) it can be deduced that the elements in the third column of $\overset{*}{X}_C \overset{*}{K}_{CD} \overset{*}{X}_D{}^T$ are 0, 0, and $\frac{w}{v_R}$. In order to find the elements in the first two columns of the matrix product, the following notation is introduced:

$$\overset{*}{K}_{CD}' = \begin{Bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{Bmatrix} = \begin{Bmatrix} \underline{k}_1{}^T \\ \underline{k}_2{}^T \\ \underline{k}_3{}^T \end{Bmatrix} \tag{N-17}$$

For i = 1, 2, or 3, $\underline{k}_i$ is a vector with components k_{i1}, k_{i2}, and k_{i3} along the r_1, r_2, and r_3 axes corresponding to time t_D.

$$\overset{*}{K}_{CD} \overset{*}{X}_D{}^T = (\overset{*}{X}_D \overset{*}{K}_{CD}{}^T)^T = \left\{ \overset{*}{X}_D (\underline{k}_1 \quad \underline{k}_2 \quad \underline{k}_3) \right\}^T$$

$$= \left\{ (\underline{k}_1)_W \quad (\underline{k}_2)_W \quad (\underline{k}_3)_W \right\}^T$$

$$= \begin{Bmatrix} (\underline{k}_1{}^T)_W \\ (\underline{k}_2{}^T)_W \\ (\underline{k}_3{}^T)_W \end{Bmatrix} = \begin{Bmatrix} k_{1\xi} & k_{1\eta} & k_{1\zeta} \\ k_{2\xi} & k_{2\eta} & k_{2\zeta} \\ k_{3\xi} & k_{3\eta} & k_{3\zeta} \end{Bmatrix} \tag{N-18}$$

$k_{i\xi}$, $k_{i\eta}$, and $k_{i\zeta}$ are the components of $\underline{k}_i$ along the ξ_D, η_D, and ζ_D axes, respectively.

$$(\underline{k}_i)_W = \begin{Bmatrix} k_{i\xi} \\ k_{i\eta} \\ k_{i\zeta} \end{Bmatrix} = \begin{Bmatrix} k_{i1} \cos \Omega_D + k_{i2} \sin \Omega_D \\ -(k_{i1} \sin \Omega_D - k_{i2} \cos \Omega_D) \cos i_D + k_{i3} \sin i_D \\ (k_{i1} \sin \Omega_D - k_{i2} \cos \Omega_D) \sin i_D + k_{i3} \cos i_D \end{Bmatrix} \tag{N-19}$$

With $\overset{*}{K}_{CD} \overset{*}{X}_D{}^T$ expressed in terms of the k-components by Equation (N-18), the next step is to obtain similar expressions for the elements of $\overset{*}{X}_C$. The elements of the third row of $\overset{*}{X}_C$ are readily derived.

$$
\begin{aligned}
\underline{w}^T &= \left\{ \overset{*}{X}_C{}^T (\underline{w})_W \right\}^T = (\underline{w}^T)_W \overset{*}{X}_C \\
&= w \left\{ 0 \quad 0 \quad 1 \right\} \overset{*}{X}_C \\
&= \left\{ \overset{*}{K}_{CD} \overset{*}{X}_D{}^T (\underline{v}_R)_W \right\}^T \\
&= v_R \left\{ 0 \quad 0 \quad 1 \right\} (\overset{*}{K}_{CD} \overset{*}{X}_D{}^T)^T
\end{aligned}
\tag{N-20}
$$

The third row of $\overset{*}{X}_C$ is

$$
\begin{aligned}
\left\{ 0 \quad 0 \quad 1 \right\} \overset{*}{X}_C &= \left\{ \sin \Omega_C \sin i_C \quad -\cos \Omega_C \sin i_C \quad \cos i_C \right\} \\
&= \frac{v_R}{w} \left\{ 0 \quad 0 \quad 1 \right\} (\overset{*}{K}_{CD} \overset{*}{X}_D{}^T)^T \\
&= \frac{v_R}{w} \left\{ k_{1\zeta} \quad k_{2\zeta} \quad k_{3\zeta} \right\}
\end{aligned}
\tag{N-21}
$$

From (N-20),

$$
w^2 = \underline{w}^T \underline{w} = v_R{}^2 (k_{1\zeta}{}^2 + k_{2\zeta}{}^2 + k_{3\zeta}{}^2) \tag{N-22}
$$

$$
\frac{w}{v_R} = (k_{1\zeta}{}^2 + k_{2\zeta}{}^2 + k_{3\zeta}{}^2)^{1/2} \tag{N-23}
$$

With the aid of (N-21) and (N-23), the entire $\overset{*}{X}_C$ matrix can be written as shown in (N-24). Only the ζ- components of the three $\underline{k}$ vectors are involved in X_C. Finally, (N-18) and (N-24) can be combined to yield the expression for $\overset{*}{X}_C \overset{*}{K}_{CD} \overset{*}{X}_D{}^T$ given by (N-25).

$$\overset{*}{X}_C = \left\{ \begin{array}{l} \dfrac{1}{(k_{1\zeta}{}^2 + k_{2\zeta}{}^2)^{1/2}} \begin{pmatrix} -k_{2\zeta} & k_{1\zeta} & 0 \end{pmatrix} \\ \hdashline \dfrac{1}{(k_{1\zeta}{}^2 + k_{2\zeta}{}^2)^{1/2} (k_{1\zeta}{}^2 + k_{2\zeta}{}^2 + k_{3\zeta}{}^2)^{1/2}} \begin{pmatrix} -k_{1\zeta} k_{3\zeta} & -k_{2\zeta} k_{3\zeta} & k_{1\zeta}{}^2 + k_{2\zeta}{}^2 \end{pmatrix} \\ \hdashline \dfrac{1}{(k_{1\zeta}{}^2 + k_{2\zeta}{}^2 + k_{3\zeta}{}^2)^{1/2}} \begin{pmatrix} k_{1\zeta} & k_{2\zeta} & k_{3\zeta} \end{pmatrix} \end{array} \right\} \tag{N-24}$$

$$\overset{*}{X}_C \overset{*}{K}_{CD} \overset{*}{X}_D{}^T = \left\{ \begin{array}{l} \dfrac{1}{(k_{1\zeta}{}^2 + k_{2\zeta}{}^2)^{1/2}} \begin{pmatrix} -k_{2\zeta} k_{1\xi} + k_{1\zeta} k_{2\xi} & -k_{2\zeta} k_{1\eta} + k_{1\zeta} k_{2\eta} & 0 \end{pmatrix} \\ \hdashline \dfrac{1}{(k_{1\zeta}{}^2 + k_{2\zeta}{}^2)^{1/2} (k_{1\zeta}{}^2 + k_{2\zeta}{}^2 + k_{3\zeta}{}^2)^{1/2}} \begin{pmatrix} -k_{3\zeta}(k_{1\zeta} k_{1\xi} + k_{2\zeta} k_{2\xi}) & -k_{3\zeta}(k_{1\zeta} k_{1\eta} + k_{2\zeta} k_{2\eta}) & \\ + k_{3\xi}(k_{1\zeta}{}^2 + k_{2\zeta}{}^2) & + k_{3\eta}(k_{1\zeta}{}^2 + k_{2\zeta}{}^2) & 0 \end{pmatrix} \\ \hdashline \dfrac{1}{(k_{1\zeta}{}^2 + k_{2\zeta}{}^2 + k_{3\zeta}{}^2)^{1/2}} \begin{pmatrix} k_{1\zeta} k_{1\xi} + k_{2\zeta} k_{2\xi} + k_{3\zeta} k_{3\xi} & k_{1\zeta} k_{1\eta} + k_{2\zeta} k_{2\eta} + k_{3\zeta} k_{3\eta} & k_{1\zeta}{}^2 + k_{2\zeta}{}^2 + k_{3\rho}{}^2 \end{pmatrix} \end{array} \right\} \tag{N-25}$$

N.6 Velocity Correction

In the critical-plane coordinate system both the VTA velocity correction vector and the miss distance vector become two-dimensional vectors. Therefore, the matrix relating the two must reduce to a 2-by-2 matrix. The characteristics of this 2-by-2 matrix are investigated in this section.

From Equation (M-10),

$$(\underline{c}_V)_W = \overset{*}{X}_C \, \underline{c}_V = - \overset{*}{X}_C \left(\overset{*}{I}_3 - \frac{\underline{w}\,\underline{w}^T}{\underline{w}^T \underline{w}} \right) \overset{*}{K}_{CD} \, \delta \underline{r}_D^-$$

$$= - \overset{*}{X}_C \left(\overset{*}{I}_3 - \frac{\overset{*}{X}_C{}^T (\underline{w})_W (\underline{w}^T)_W \overset{*}{X}_C}{w^2} \right) \overset{*}{K}_{CD} \overset{*}{X}_D{}^T (\delta \underline{r}_D^-)_W$$

$$= - \left(\overset{*}{I}_3 - \frac{(\underline{w})_W (\underline{w}^T)_W}{w^2} \right) \overset{*}{X}_C \overset{*}{K}_{CD} \overset{*}{X}_D{}^T (\delta \underline{r}_D^-)_W$$

$$= - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \overset{*}{X}_C \overset{*}{K}_{CD} \overset{*}{X}_D{}^T (\delta \underline{r}_D^-)_W \tag{N-26}$$

When (N-25) is substituted into (N-26), the equation for the VTA correction may be written as

$$\underline{c}_W = \overset{*}{Y} (\delta \underline{\rho}^-)_W \tag{N-27}$$

The two-dimensional correction vector $\underline{c}_W$ consists of the components of $\underline{c}_V$ in the ξ_C and η_C directions.

$$\underline{c}_W = \begin{Bmatrix} c_\xi \\ c_\eta \end{Bmatrix} \tag{N-28}$$

$(\delta \underline{\rho}^-)_W$ consists of the components of the miss distance vector $\delta \underline{\rho}^-$ in the ξ_D and η_D directions.

$$(\delta \underline{\rho}^-)_W = \begin{Bmatrix} \delta \xi_D^- \\ \delta \eta_D^- \end{Bmatrix} \tag{N-29}$$

$\overset{*}{Y}$ is the negative of the 2-by-2 sub-matrix made up of the elements of the first two rows and the first two columns of $\overset{*}{X}_C \overset{*}{K}_{CD} \overset{*}{X}_D{}^T$. This is indicated in Equation (N-30).

$$\overset{*}{Y} = \frac{1}{(k_{1\zeta}^2 + k_{2\zeta}^2)^{1/2}} \left\{ \begin{array}{c} \begin{pmatrix} k_{2\zeta} k_{1\xi} - k_{1\zeta} k_{2\xi} & k_{2\zeta} k_{1\eta} - k_{1\zeta} k_{2\eta} \end{pmatrix} \\ \hline \dfrac{1}{(k_{1\zeta}^2 + k_{2\zeta}^2 + k_{3\zeta}^2)^{1/2}} \begin{pmatrix} k_{3\zeta}(k_{1\zeta} k_{1\xi} + k_{2\zeta} k_{2\xi}) & k_{3\zeta}(k_{1\zeta} k_{1\eta} + k_{2\zeta} k_{2\eta}) \\ - k_{3\xi}(k_{1\zeta}^2 + k_{2\zeta}^2) & - k_{3\eta}(k_{1\zeta}^2 + k_{2\zeta}^2) \end{pmatrix} \end{array} \right\} \tag{N-30}$$

The terms in the first row of $\overset{*}{Y}$ are independent of $\underline{k}_3$; consequently, the component of the correction along the line of nodes is not affected by the elements in the third row of $\overset{*}{K}_{CD}$.

The matrix $\overset{*}{Y}$ may be simplified by introducing the angles α and β, which are defined by the following trigonometric relationships:

$$\sin \alpha = \frac{k_{2\zeta}}{(k_{1\zeta}^2 + k_{2\zeta}^2)^{1/2}} \qquad \cos \alpha = \frac{k_{1\zeta}}{(k_{1\zeta}^2 + k_{2\zeta}^2)^{1/2}} \tag{N-31}$$

$$\sin \beta = \frac{k_{3\zeta}}{(k_{1\zeta}^2 + k_{2\zeta}^2 + k_{3\zeta}^2)^{1/2}} \qquad \cos \beta = \frac{(k_{1\zeta}^2 + k_{2\zeta}^2)^{1/2}}{(k_{1\zeta}^2 + k_{2\zeta}^2 + k_{3\zeta}^2)^{1/2}} \tag{N-32}$$

Then $\overset{*}{Y}$ becomes

$$\overset{*}{Y} = \begin{Bmatrix} k_{1\xi} \sin \alpha - k_{2\xi} \cos \alpha & k_{1\eta} \sin \alpha - k_{2\eta} \cos \alpha \\ (k_{1\xi} \cos \alpha + k_{2\xi} \sin \alpha) \sin \beta & (k_{1\eta} \cos \alpha + k_{2\eta} \sin \alpha) \sin \beta \\ - k_{3\xi} \cos \beta & - k_{3\eta} \cos \beta \end{Bmatrix} \tag{N-33}$$

The ζ-components of the $\underline{k}$-vectors no longer appear explicitly. Only the ξ-components appear in the first column of $\overset{*}{Y}$; only the η-components appear in the second column. These observations may be related to the velocity correction equation, (N-27), by stating that the coefficients of $\delta\xi_D^-$ contain only the ξ-components of the $\underline{k}$-vectors, while the coefficients of $\delta\eta_D^-$ contain only the η-components.

N.7 Selection of Time of Correction

For a known miss distance vector, the optimum time of correction is defined as that time for which the magnitude of the required correction is a minimum.

Let the two-dimensional miss distance vector $(\delta\underline{\rho}^-)_W$ be represented by a magnitude $\delta\rho^-$ and a phase angle ψ. ψ is the angle in the $\xi_D - \eta_D$ plane between the ξ_D-axis and $\delta\underline{\rho}^-$. From (N-29),

$$(\delta\underline{\rho}^-)_W = \begin{Bmatrix} \delta\xi_D^- \\ \delta\eta_D^- \end{Bmatrix} = (\delta\rho^-) \begin{Bmatrix} \cos\psi \\ \sin\psi \end{Bmatrix} \tag{N-34}$$

The square of the magnitude of the VTA correction is

$$\begin{aligned} c_V^2 &= \underline{c}_W^T \, \underline{c}_W = (\delta\underline{\rho}^-)_W^T \, \overset{*}{Y}{}^T \overset{*}{Y} \, (\delta\underline{\rho}^-)_W \\ &= (\delta\rho^-)^2 \begin{Bmatrix} \cos\psi & \sin\psi \end{Bmatrix} \overset{*}{Y}{}^T \overset{*}{Y} \begin{Bmatrix} \cos\psi \\ \sin\psi \end{Bmatrix} \end{aligned} \tag{N-35}$$

It is apparent that c_V varies linearly with $\delta\rho^-$, but its variation with ψ is non-linear. Its variation with t_C is also non-linear due to the dependence of $\overset{*}{Y}$ on t_C.

The procedure to be followed in determining the optimum correction time is to use Equation (N-35) to plot $c_V/\delta\rho^-$ as a function of t_C for a number of fixed values of ψ. The minimum value of $c_V/\delta\rho^-$ for a given ψ occurs at the optimum correction time for that ψ. Finally, cross-plots are made of $(c_V/\delta\rho^-)_{min}$ and $t_{C\ opt}$ versus ψ. The latter curve defines the optimum correction time as a function of the single parameter ψ of the space vehicle's variant path.

Although ψ can have any value between 0° and 360°, only values between 0° and 180° need be used in the plots, since an increment of 180° in ψ reverses the direction of $\underline{c}_V$ but has no effect on its magnitude.

N.8 Application to Two-Body Reference Trajectories

For two-body reference trajectories the elements k_{13}, k_{23}, k_{31}, and k_{32} of $\overset{*}{K}_{CD}$ are all zero. From Equation (N-19), the three $(\underline{k}_i)_W$ vectors are given by

$$(\underline{k}_1)_W = \begin{Bmatrix} k_{1\xi} \\ k_{1\eta} \\ k_{1\zeta} \end{Bmatrix} = \begin{Bmatrix} k_{11} \cos \Omega_D + k_{12} \sin \Omega_D \\ -(k_{11} \sin \Omega_D - k_{12} \cos \Omega_D) \cos i_D \\ (k_{11} \sin \Omega_D - k_{12} \cos \Omega_D) \sin i_D \end{Bmatrix} \qquad \text{(N-36)}$$

$$(\underline{k}_2)_W = \begin{Bmatrix} k_{2\xi} \\ k_{2\eta} \\ k_{2\zeta} \end{Bmatrix} = \begin{Bmatrix} k_{21} \cos \Omega_D + k_{22} \sin \Omega_D \\ -(k_{21} \sin \Omega_D - k_{22} \cos \Omega_D) \cos i_D \\ (k_{21} \sin \Omega_D - k_{22} \cos \Omega_D) \sin i_D \end{Bmatrix} \qquad \text{(N-37)}$$

$$(\underline{k}_3)_W = \left\{ \begin{matrix} k_{3\xi} \\ k_{3\eta} \\ k_{3\zeta} \end{matrix} \right\} = \left\{ \begin{matrix} 0 \\ k_{33} \sin i_D \\ k_{33} \cos i_D \end{matrix} \right\} \tag{N-38}$$

Then,

$$\tan i_D = -\frac{k_{1\zeta}}{k_{1\eta}} = -\frac{k_{2\zeta}}{k_{2\eta}} = \frac{k_{3\eta}}{k_{3\zeta}} \tag{N-39}$$

The upper right-hand element of $\overset{*}{Y}$ in Equation (N-30) becomes zero, and the matrix may be written

$$\overset{*}{Y} = \left\{ \begin{matrix} k_{1\xi} \sin \alpha - k_{2\xi} \cos \alpha & 0 \\ (k_{1\xi} \cos \alpha + k_{2\xi} \sin \alpha) \sin \beta & \begin{matrix} (k_{1\eta} \cos \alpha + k_{2\eta} \sin \alpha) \sin \beta \\ - k_{3\eta} \cos \beta \end{matrix} \end{matrix} \right\} \tag{N-40}$$

The triangular form taken by $\overset{*}{Y}$ for two-body reference trajectories indicates that for such trajectories the component of the correction in the direction of the line of nodes at t_C depends on only that component of the miss distance which lies in the direction of the line of nodes at t_D. This partial uncoupling effect is somewhat surprising; it was not anticipated when the critical-plane coordinate system was originally introduced.

It is of some interest to express the elements of $\overset{*}{Y}$ in (N-40) in terms of the fundamental parameters, namely, the elements of $\overset{*}{K}_{CD}$ and angles Ω_D and i_D. Let y_{ij} be the element in the i-the row and j-th column of $\overset{*}{Y}$.

$$y_{11} = k_{1\xi} \sin \alpha - k_{2\xi} \cos \alpha$$

$$= \frac{k_{1\xi}\, k_{2\zeta} - k_{2\xi}\, k_{1\zeta}}{(k_{1\zeta}^{2} + k_{2\zeta}^{2})^{1/2}}$$

$$= \frac{1}{A} \left[(k_{11} \cos \Omega_D + k_{12} \sin \Omega_D)(k_{21} \sin \Omega_D - k_{22} \cos \Omega_D) \right.$$

$$\left. - (k_{21} \cos \Omega_D + k_{22} \sin \Omega_D)(k_{11} \sin \Omega_D - k_{12} \cos \Omega_D) \right]$$

$$= \frac{1}{A} (k_{12}\, k_{21} - k_{11}\, k_{22}) \tag{N-41}$$

where

$$A = \frac{(k_{1\zeta}^{2} + k_{2\zeta}^{2})^{1/2}}{\sin i_D}$$

$$= \left[(k_{11} \sin \Omega_D - k_{12} \cos \Omega_D)^2 + (k_{21} \sin \Omega_D - k_{22} \cos \Omega_D)^2 \right]^{1/2} \tag{N-42}$$

$$y_{21} = (k_{1\xi} \cos \alpha + k_{2\xi} \sin \alpha) \sin \beta$$

$$= \frac{k_{3\zeta} (k_{1\xi}\, k_{1\zeta} + k_{2\xi}\, k_{2\zeta})}{(k_{1\zeta}^{2} + k_{2\zeta}^{2})^{1/2} (k_{1\zeta}^{2} + k_{2\zeta}^{2} + k_{3\zeta}^{2})^{1/2}}$$

$$= \frac{k_{33}\, v_R \cos i_D}{Aw} \left[(k_{11} \cos \Omega_D + k_{12} \sin \Omega_D)(k_{11} \sin \Omega_D - k_{12} \cos \Omega_D) \right.$$

$$\left. + (k_{21} \cos \Omega_D + k_{22} \sin \Omega_D)(k_{21} \sin \Omega_D - k_{22} \cos \Omega_D) \right]$$

$$= \frac{k_{33}\, v_R \cos i_D}{Aw} \left[(k_{11}^{2} - k_{12}^{2} + k_{21}^{2} - k_{22}^{2}) \sin \Omega_D \cos \Omega_D \right.$$

$$\left. + (k_{11}\, k_{12} + k_{21}\, k_{22})(\sin^2 \Omega_D - \cos^2 \Omega_D) \right] \tag{N-43}$$

From (N-23),

$$\frac{v_R}{w} = (k_{1\zeta}^2 + k_{2\zeta}^2 + k_{3\zeta}^2)^{-1/2}$$

$$= \left\{ [(k_{11} \sin \Omega_D - k_{12} \cos \Omega_D)^2 + (k_{21} \sin \Omega_D - k_{22} \cos \Omega_D)^2] \sin^2 i_D \right.$$

$$\left. + k_{33}^2 \cos^2 i_D \right\}^{-1/2}$$

$$= (A^2 \sin^2 i_D + k_{33}^2 \cos^2 i_D)^{-1/2} \qquad \text{(N-44)}$$

$$y_{22} = (k_{1\eta} \cos \alpha + k_{2\eta} \sin \alpha) \sin\beta - k_{3\eta} \cos \beta$$

$$= k_{1\eta} \left(\cos \alpha + \frac{k_{2\zeta}}{k_{1\zeta}} \sin \alpha \right) \sin \beta - k_{3\eta} \cos \beta$$

$$= \frac{k_{1\eta}}{(k_{1\zeta}^2 + k_{2\zeta}^2)^{1/2}} \left(k_{1\zeta} + \frac{k_{2\zeta}^2}{k_{1\zeta}} \right) \sin \beta - k_{3\eta} \cos \beta$$

$$= \frac{k_{1\eta}}{k_{1\zeta}} (k_{1\zeta}^2 + k_{2\zeta}^2)^{1/2} \sin \beta - k_{3\eta} \cos \beta$$

$$= \frac{v_R}{w} (k_{1\zeta}^2 + k_{2\zeta}^2)^{1/2} \left(\frac{k_{1\eta} k_{3\zeta}}{k_{1\zeta}} - k_{3\eta} \right)$$

$$= - \frac{v_R}{w} k_{33} A \qquad \text{(N-45)}$$

$\overset{*}{Y}$ for two-body reference trajectories is then

$$\overset{*}{Y} = \begin{Bmatrix} \frac{1}{A}(k_{12}k_{21} - k_{11}k_{22}) & 0 \\ \\ \frac{1}{A}\frac{v_R}{w}k_{33}\cos i_D[(k_{11}^2 - k_{12}^2 & \\ + k_{21}^2 - k_{22}^2)\sin\Omega_D\cos\Omega_D & -A\frac{v_R}{w}k_{33} \\ + (k_{11}k_{12} + k_{21}k_{22})(\sin^2\Omega_D & \\ - \cos^2\Omega_D)] & \end{Bmatrix} \tag{N-46}$$

N.9 Evaluation of Parameters

The two fundamental parameters used in the analysis contained in this appendix are the orientation angles Ω_D and i_D. They can be evaluated by means of Equations (N-11), (N-12), and (N-14).

$$\underline{v}_R = \begin{Bmatrix} v_{R_1} \\ v_{R_2} \\ v_{R_3} \end{Bmatrix} = \overset{*}{X}_D{}^T (\underline{v}_R)_W = v_R \begin{Bmatrix} \sin\Omega_D \sin i_D \\ -\cos\Omega_D \sin i_D \\ \cos i_D \end{Bmatrix} \tag{N-47}$$

The components of $\underline{v}_R$ along the r_1, r_2, and r_3 axes are known for the specified reference trajectory. The desired angles are computed from these components.

$$\Omega_D = -\arctan\left(\frac{v_{R_1}}{v_{R_2}}\right) \tag{N-48}$$

$$i_D = \arccos\left(\frac{v_{R_3}}{v_R}\right) \tag{N-49}$$

The quadrant location of each of the two angles is determined by stipulating that each lies in the range between 0 and π radians.

When Ω_D and i_D have been evaluated, the transformation matrix $\overset{*}{X}_D$ is computed from (N-11). This matrix and matrix $\overset{*}{K}_{CD}$, the evaluation of which has been discussed in Section L.6, are used to calculate the components of the three $\underline{k}$ vectors in the critical-plane coordinate system.

From (N-18),

$$\begin{Bmatrix} k_{1\xi} & k_{1\eta} & k_{1\zeta} \\ k_{2\xi} & k_{2\eta} & k_{2\zeta} \\ k_{3\xi} & k_{3\eta} & k_{3\zeta} \end{Bmatrix} = \overset{*}{K}_{CD} \overset{*}{X}_D{}^T \qquad \text{(N-50)}$$

The elements of $\overset{*}{Y}$ can then be determined from (N-30) or from (N-33) used in conjunction with (N-31) and (N-32).

An estimate of the components of $\delta \underline{r}_D^-$ along the r_1, r_2, and r_3 axes is assumed to be available. Matrix $\overset{*}{X}_D$ transforms $\delta \underline{r}_D^-$ into its components along ξ_D, η_D, and ζ_D axes.

$$\overset{*}{X}_D \, \delta \underline{r}_D^- = (\delta \underline{r}_D^-)_W = \begin{Bmatrix} \delta \xi_D^- \\ \delta \eta_D^- \\ \delta \zeta_D^- \end{Bmatrix} = \begin{Bmatrix} (\delta \underline{\rho}^-)_W \\ \delta \zeta_D^- \end{Bmatrix} \qquad \text{(N-51)}$$

$\overset{*}{Y}$ and $(\delta \underline{\rho}^-)_W$ are used to compute the components of the VTA velocity correction along critical-plane coordinate axes by means of Equation (N-27). The components of the correction along the r_1, r_2, and r_3 axes are obtained from the equation

$$\underline{c}_V = \overset{*}{X}_C{}^T \, \underline{c}_W \qquad \text{(N-52)}$$

Equation (N-24) can be used to compute the elements of $\overset{*}{X}_C$ from the ζ-components of the three $\underline{k}$ vectors, which have already been computed.

An alternative method of computing the elements of $\overset{*}{X}_C$, which serves as a partial check of the computing procedure, is by means of Equation (N-13).

$$\underline{w} = \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} = \overset{*}{K}_{CD} \underline{v}_R = \overset{*}{X}_C{}^T (\underline{w})_W = w \begin{Bmatrix} \sin \Omega_C \sin i_C \\ - \cos \Omega_C \sin i_C \\ \cos i_C \end{Bmatrix} \qquad \text{(N-53)}$$

The components w_1, w_2, and w_3 are determined, and the angles Ω_C and i_C are expressed in terms of these components.

$$\Omega_C = - \arctan \left(\frac{w_1}{w_2} \right) \qquad \text{(N-54)}$$

$$i_C = \arccos \left(\frac{w_3}{w} \right) \qquad \text{(N-55)}$$

Both angles are restricted to the range 0 to π radians. The elements of $\overset{*}{X}_C$ can be found from the angles by the use of (N-11).

The angle ψ, used in determining the optimum correction time, can be computed from Equation (N-34).

$$\psi = \arctan \left(\frac{\delta \eta_D^-}{\delta \xi_D^-} \right) \qquad \text{(N-56)}$$

It has already been pointed out in Section N.7 that only values of ψ in the range 0 to π radians need be considered in the procedure for optimizing the time of correction; consequently, the angles computed by means of (N-56) can be restricted to that range.

The computational procedure is simplified in the case of two-body reference trajectories. The matrix $\overset{*}{Y}$ can be found from Ω_D, i_D, and the elements of $\overset{*}{K}_{CD}$ by the use of Equations (N-42), (N-44), and (N-46). If the alternate form is used to compute Ω_C and i_C, there is no need to transform the $\underline{k}$ vectors into the critical-plane coordinate system.

APPENDIX O

SINGULARITIES IN THE MATRIX SOLUTION FOR ELLIPTICAL TRAJECTORIES

O. 1 Summary

In the analytical development for elliptical trajectories presented in Appendix K, it has been shown that the variations in the orbital elements, represented by the vector $\delta \underline{e}$, can be expressed in terms of two position variations $\delta \underline{r}_i$ and $\delta \underline{r}_j$. The 6-by-6 matrix relating $\delta \underline{e}$ to $\delta \underline{r}_i$ and $\delta \underline{r}_j$ is obtained by inverting the 6-by-6 matrix through which $\delta \underline{r}_i$ and $\delta \underline{r}_j$ are expressed in terms of $\delta \underline{e}$. However, the latter matrix becomes singular and hence cannot be inverted, for three different types of combinations of t_i and t_j. These three types are

(1) $\quad t_j - t_i = N P$

(2) $\quad f_j - f_i = (2 N - 1) \pi$

(3) $\quad X = 0$

where N is a positive integer, P is the period of the reference trajectory, and the factor X is defined by Eqs. (K-14), (K-15), and (K-17).

This appendix examines the mathematical consequences of the singularities and interprets them physically. Explanation of the first two types of singularities is relatively simple. The third type is more subtle; Lambert's theorem, in classical celestial mechanics theory, is used in its interpretation.

If the time of midcourse correction is related to the nominal time of arrival in such a manner that any one of the singularity conditions is satisfied, no finite FTA velocity correction can be computed.

The use of VTA guidance tends to mitigate the effect of the singularities. For a correction time corresponding to either the second or the third type of singularity, a VTA correction of finite magnitude can be determined. However, if the correction time meets the condition for the first type of singularity, the magnitude of the computed correction is infinite even in VTA guidance.

O. 2 Preliminary Remarks

When the analytic solution of the guidance problem for elliptical trajectories was first obtained, it became a matter of considerable interest to find a physical explanation for the various singularities. Singularities of the first two types had already been recognized by Laning and Battin from their numerical studies. (See Pages 201 and 202 of Reference (5)). There is no indication of the singularity at $X = 0$ in any of the technical literature that has been reviewed.

The verbal disclosure of the $X = 0$ singularity was initially greeted with a degree of skepticism, because, unlike the other types, it did not have a physical interpretation that was immediately apparent. Much of the skepticism was allayed when evidence of the existence of this singularity was found in the computer data used in Reference (5). It was not until some time later that the mathematical connection between the singularity at $X = 0$ and the minimum point on the time-of-flight curve was proved and a physical explanation of the singularity was presented.

O. 3 The Singular Matrix

The position variations $\delta \underline{r}_i$ and $\delta \underline{r}_j$ are related to $\delta \underline{e}$ by the equation

$$\begin{Bmatrix} \delta \underline{r}_i \\ \delta \underline{r}_j \end{Bmatrix} = \begin{Bmatrix} \overset{*}{F}_i \\ \overset{*}{F}_j \end{Bmatrix} \delta \underline{e} = \overset{*}{A}_{ij} \, \delta \underline{e} \tag{O-1}$$

An analytic expression for the 3-by-6 matrix $\overset{*}{F}_j$ is given by Eq. (K-31). $\delta \underline{e}$ is defined by Eq. (K-1).

It is the 6-by-6 matrix $\overset{*}{A}_{ij}$, comprised of $\overset{*}{F}_i$ and $\overset{*}{F}_j$, that becomes singular under the conditions specified in Section O. 1. When this matrix is singular, the six components comprising $\delta \underline{r}_i$ and $\delta \underline{r}_j$ are not linearly independent, and consequently all the elements of $\delta \underline{e}$ cannot be determined uniquely.

The non-zero elements of $\overset{*}{A}_{ij}$ may be grouped into two sub-matrices, the first of which is the 4-by-4 matrix pertaining to motion in the plane of the reference trajectory and the second of which is the 2-by-2 matrix pertaining to motion parallel to the z-axis. A singularity may occur in either or both sub-matrices. The first four components of $\delta \underline{e}$ are used in conjunction with the 4-by-4 sub-matrix; the last two components of $\delta \underline{e}$ are used in conjunction with the 2-by-2 sub-matrix.

Because the two types of motion are uncoupled, if a singularity occurs only in the 4-by-4 sub-matrix, the last two components of $\delta \underline{e}$ can still be evaluated; conversely, if a singularity occurs only in the 2-by-2 sub-matrix, the first four elements of the $\delta \underline{e}$ can still be evaluated.

O. 4 Mathematical Study of Singularities at $(t_j - t_i) = N P$

The singularities for which $(t_j - t_i) = N P$ will be examined first. Even without mathematical analysis, it is intuitively reasonable to expect that two position variations obtained at times that are an exact number of reference periods apart will bear some relation to each other and hence will not be independent.

In one circuit about the attractive focus, the change in each of the three anomalies - real, eccentric, and mean - is exactly 2π radians. Thus, when $(t_j - t_i) = N P$, the eccentric anomaly difference is $2N\pi$ radians. It may be deduced from Eq. (K-31) that the rank of matrix $\overset{*}{A}_{ij}$ is reduced to four when $(t_j - t_i) = N P$. The rank of the 4-by-4 sub-matrix is reduced to three, and the rank of the 2-by-2 sub-matrix is reduced to one.

Eq. (K-31) may also be used to show the relation between $\delta \underline{r}_j$ and $\delta \underline{r}_i$ for the singularity condition.

$$\begin{Bmatrix} \delta p_j \\ \delta q_j \\ \delta z_j \end{Bmatrix} = \begin{Bmatrix} \delta p_i \\ \delta q_i \\ \delta z_i \end{Bmatrix} - \begin{Bmatrix} 0 \\ \frac{3}{2} \frac{(1 + e \cos E_i)^{1/2}}{(1 - e \cos E_i)^{1/2}} \; 2 N \pi \; \delta a \\ 0 \end{Bmatrix} \tag{O-2}$$

With the aid of Eq. (B-69), (O-2) may be expressed in terms of the nominal orbital velocity v_i and the nominal period P.

$$\begin{Bmatrix} \delta p_j \\ \delta q_j \\ \delta z_j \end{Bmatrix} = \begin{Bmatrix} \delta p_i \\ \delta q_i \\ \delta z_i \end{Bmatrix} - \begin{Bmatrix} 0 \\ \frac{3}{2} N P v_i \frac{\delta a}{a} \\ 0 \end{Bmatrix} \tag{O-3}$$

Eq. (O-3) can be solved for the third component of $\delta \underline{e}$.

$$\frac{1}{2} \frac{\delta a}{a} = - \frac{\delta q_j - \delta q_i}{3 N P v_i} \tag{O-4}$$

It is interesting that this unique solution for $\frac{1}{2} \frac{\delta a}{a}$ exists despite the fact that $\overset{*}{A}_{ij}$ contains singularities in both of its non-zero sub-matrices.

If t_i is associated with t_C, the time at which the correction is to be applied, and if t_j is associated with t_D, the nominal time of arrival, Eq. (O-3) becomes

$$\begin{Bmatrix} \delta p_D \\ \delta q_D \\ \delta z_D \end{Bmatrix} = \begin{Bmatrix} \delta p_C \\ \delta q_C \\ \delta z_C \end{Bmatrix} - \begin{Bmatrix} 0 \\ \frac{3}{2} N P v_C \frac{\delta a}{a} \\ 0 \end{Bmatrix} \tag{O-5}$$

If a position variation exists in either the p_C or the z_C direction at time t_C, that same position variation will exist at time t_D irrespective of the nature of the path traversed by the vehicle in the N circuits between t_C and t_D. Linear theory does not permit the computation of a velocity correction which, if applied at $t = t_C$, will cause the position variations δp and δz to be reduced to zero when $t = t_D$.

In the special case when $\delta p_C = 0 = \delta z_C$ and $\delta q_C \neq 0$, it is possible to compute a velocity correction $\underline{c}_F$ which will enable the vehicle to arrive at the desired destination at the proper time. The correction required to reduce the predicted value of δq_D to zero is such that

$$\left(\frac{\delta a}{a}\right)^{+} = \frac{2\ \delta q_C^{-}}{3\ N\ P\ v_C} = \frac{2\ \delta q_C^{-}}{3\ N\ P\ v_D} \tag{O-6}$$

The + and - superscripts have been added to distinguish characteristics of the corrected path from characteristics of the original path. From Eqs.(O-5) and (O-6),

$$\left(\frac{\delta a}{a}\right)^{+} = \frac{2\ \delta q_D^{-}}{3\ N\ P\ v_D} + \left(\frac{\delta a}{a}\right)^{-} \tag{O-7}$$

so that the change in $\frac{\delta a}{a}$ to be provided by the correction is

$$\left(\frac{\delta a}{a}\right)^{+} - \left(\frac{\delta a}{a}\right)^{-} = \frac{2\ \delta q_D^{-}}{3\ N\ P\ v_D} \tag{O-8}$$

The velocity correction itself for this special case may be found from Eq. (K-48) and (L-1).

$$\underline{c}_F = -\overset{*}{K}_{CD}\ \delta \underline{r}_D^{-} = \frac{n\ (1 - e \cos E_C)\ (\cos E_M + e \cos E_P)}{2\ (1 + e \cos E_C)\ X}\ (\delta q_D^{-})\ \underline{u}_{q_C} \tag{O-9}$$

From Eqs. (K-14), (K-15), and (K-17), with $E_D - E_C = 2 N \pi$,

$$E_P = \frac{1}{2}(E_D + E_C) = E_C + N\pi = E_D - N\pi \tag{O-10}$$

$$E_M = \frac{1}{2}(E_D - E_C) = N\pi \tag{O-11}$$

$$\begin{aligned} X &= (3 E_M - e \sin E_M \cos E_P)(\cos E_M + e \cos E_P) - 4 \sin E_M \\ &= 3 N\pi (\cos E_M + e \cos E_P) \end{aligned} \tag{O-12}$$

Eq. (B-62),(B-69), and (O-12) are used to simplify the expression for $\underline{c}_F$ given by Eq. (O-9).

$$\underline{c}_F = \frac{\mu}{3 N P a v_C^2} (\delta q_D^-) \underline{u}_{q_C} = \frac{\mu}{3 N P a v_D^2} (\delta q_D^-) \underline{u}_{q_D} \tag{O-13}$$

For this special case the correction is in the direction of the nominal orbital velocity, and its magnitude is inversely proportional to the square of the nominal orbital velocity.

O.5 Physical Interpretation of Singularities at $(t_j - t_i) = N P$

The physical interpretation of the singularities at $(t_j - t_i) = N P$ will be treated in two distinct phases. In the first phase a reference trajectory is assumed, and the interpretation is based on linear perturbation theory. The second phase is more general; there is no reference trajectory and no requirement for linearization.

For the first phase, consider a vehicle traveling in an elliptical orbit which differs only slightly from a known reference ellipse. At time t_i the vehicle's position variation with respect to the reference ellipse is determined from measurements, and at time t_j, which is exactly N reference periods later than t_i, the position variation is again determined.

Let P be the period of the reference orbit and P' the period of the actual orbit. If P' = P, it is obvious that $\delta \underline{r}_j$ must be identical with $\delta \underline{r}_i$. Any difference between $\delta \underline{r}_j$ and $\delta \underline{r}_i$ must be proportional to $\delta P = (P' - P)$ and to N. $\delta \underline{r}_j$ may be expressed as follows:

$$\delta \underline{r}_j = \delta \underline{r}_i - (\underline{v}_j + \delta \underline{v}_j)\, N\, \delta P \tag{O-14}$$

The minus sign is due to the fact that an increase in the period causes a lag in the vehicle's position.

Since $\delta \underline{v}_j$ and δP are both small quantities, linear theory reduces Eq. (O-14) to

$$\begin{aligned} \delta \underline{r}_j &= \delta \underline{r}_i - N\, \underline{v}_j\, \delta P \\ &= \delta \underline{r}_i - N\, v_j\, \delta P\, \underline{u}_{q_j} \\ &= \delta \underline{r}_i - N\, v_i\, \delta P\, \underline{u}_{q_i} \end{aligned} \tag{O-15}$$

Kepler's third law states that P^2 is proportional to a^3. From this law it follows that

$$\frac{\delta P}{P} = \frac{3}{2}\, \frac{\delta a}{a} \tag{O-16}$$

Therefore,

$$\delta \underline{r}_j = \delta \underline{r}_i - \frac{3}{2}\, N\, P\, v_i\, \frac{\delta a}{a}\, \underline{u}_{q_i} \tag{O-17}$$

Eq. (O-17) is exactly the same relationship that was previously obtained as Eq. (O-3).

The foregoing discussion pertains to the problem of the determination of orbital elements from two position fixes. It can be directly related to the problem of applying a velocity correction at t_C which nulls the position variation at t_D. The analysis is the same as that presented in Section O. 4. Only if the predicted position variation at t_D is in the q_D-direction can a finite velocity correction be computed by linear theory. The computed correction for that case is given by Eq. (O-13).

For the second phase of the physical interpretation, a body is assumed to be moving in an elliptical path about an attractive focus. There is no a priori knowledge of the body's trajectory except for the fact that it is an ellipse. At time t_i the body's position relative to the focus is measured. At time t_j a second set of measurements indicates that the body's relative position is exactly the same as it was at t_i. In the interval between t_i and t_j the body has completed N circuits about the focus.

In this example the observed data consist of the times t_i and t_j, the integer N, and the three components of position.

From t_i, t_j, and N, it is possible to compute the period P, the mean angular motion n, and the semi-major axis a.

$$P = \frac{t_j - t_j}{N} \tag{O-18}$$

$$n = \frac{2\pi}{P} \tag{O-19}$$

$$a = \left(\frac{\mu}{n^2}\right)^{1/3} \tag{O-20}$$

The semi-major axis is the only one of the six orbital elements that can be obtained from the available data; all the others are indeterminate. Eq. (O-20) corresponds to Eq. (O-4) in the development based on linear theory.

Now suppose that a space vehicle is in an elliptical orbit around the sun. At time t_C, when the vehicle is at point C, a velocity correction is to be applied such that the new orbit will enable the vehicle to reach the desired destination point D at time t_D. The points C and D are relatively close to each other compared to the distance of either from the sun. The time interval $(t_D - t_C)$ is approximately N times the period of the vehicle's original orbit.

Regardless of the orientation of the line C D, it is always possible to find a new elliptical orbit which will enable the vehicle to reach D at the proper time. The plane of the new orbit must contain vectors $\underline{r}_C$ and $\underline{r}_D$, and its semi-major axis is determined by the required time interval and the number of circuits to be made between C and D.

If C D is parallel to $\underline{v}_C^-$, the velocity vector at t_C before the correction, the new orbit will closely resemble the old. All that is required is a small change in the period, which in turn causes a small change in the semi-major axis. Thus, the velocity correction itself is small in magnitude and can be computed from the linear theory. This situation is represented by the points C_1 and D in Fig. O. 1.

On the other hand, if point C has any arbitrary position in the vicinity of D, the required new orbit will in general differ drastically from the original orbit, and the velocity correction will be so large in magnitude that it cannot be determined from the linear theory.

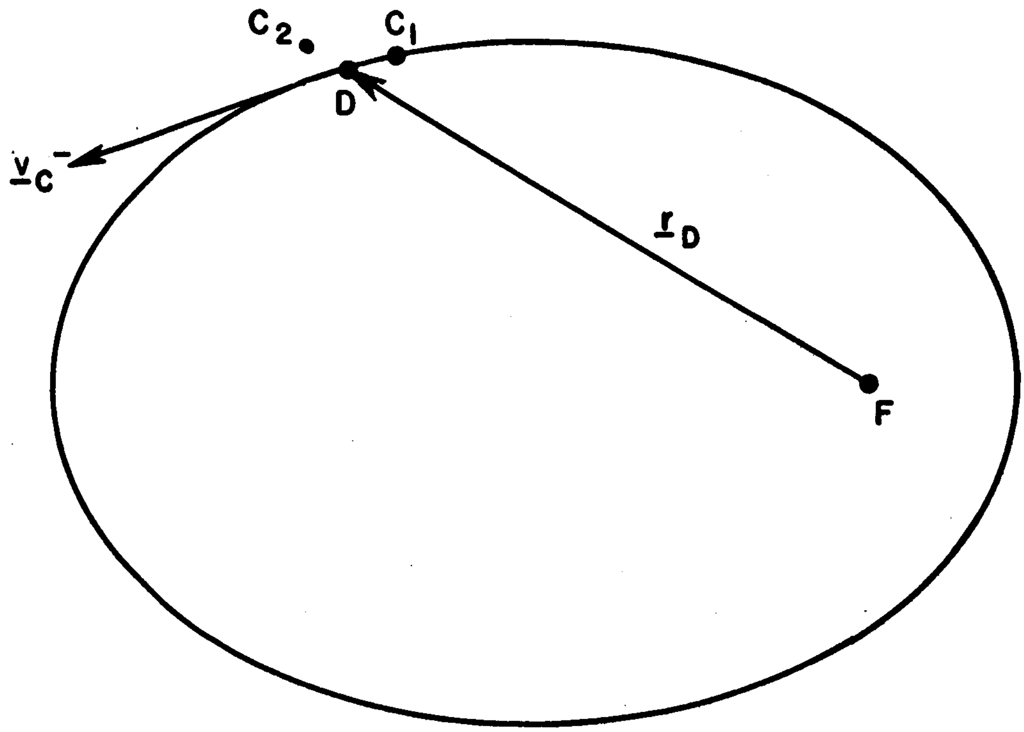
Two special cases serve to illustrate this point. In the first, the vehicle is situated at C_2 in Fig. O. 1. C_2 lies along the radial line connecting the focus (sun) with D. In this case, the new path is a rectilinear ellipse; i. e., a straight line of finite length. Obviously, the velocity correction required to change from an orbit such as the one indicated in the sketch to a rectilinear ellipse is sizable.

In the second special case C D is parallel to the z-axis; i. e., C is directly above or below D (out of the plane of the paper) in Fig. O. 1. The new path is then an ellipse in the r_D-z plane. If the distance C D is small, the velocity vector immediately after the correction, $\underline{v}_C^+$, is perpendicular to position vector $\underline{r}_C$; therefore, F C lies along the line of apsides of the new trajectory, and C is either at perihelion or at aphelion. It is clear that the magnitude of the correction required to rotate the trajectory plane through 90° is beyond the scope of the linear theory.

O. 6 Mathematical Study of Singularities at $(f_j - f_i) = (2N - 1)\pi$

When $(f_j - f_i) = (2N - 1)\pi$, the rank of matrix $\overset{*}{A}_{ij}$ is reduced to five. The rank of the 4-by-4 sub-matrix is unchanged; the rank of the 2-by-2 sub-matrix becomes one.

Since the 4-by-4 sub-matrix is not singular under these conditions, the four components of $\delta \underline{e}$ relating to motion in the reference



F – attractive focus (sun)

D – destination point

C_1, C_2 – possible vehicle positions at time of correction

$\underline{v}_C^-$ – vehicle velocity vector just prior to application of correction

$\underline{r}_D$ – vehicle position vector at destination

t_C – time of correction

t_D – time of arrival at destination

P – nominal period

Figure O.1 Special Cases of Vehicle Position at Time of Correction for Singularities at $t_D - t_C = NP$

trajectory plane can be determined from $\delta \underline{r}_i$ and $\delta \underline{r}_j$ even though $\overset{*}{A}_{ij}$ is singular.

The dependence of δz_j and δz_i is made apparent by use of Eq. (H-15).

$$\frac{\delta z_i}{r_i} = \delta i \sin (f_i - \delta \Omega) \qquad \text{(O-21)}$$

With $f_j = f_i + (2N - 1)\pi$,

$$\frac{\delta z_j}{r_j} = -\delta i \sin (f_i - \delta \Omega) \qquad \text{(O-22)}$$

Then,

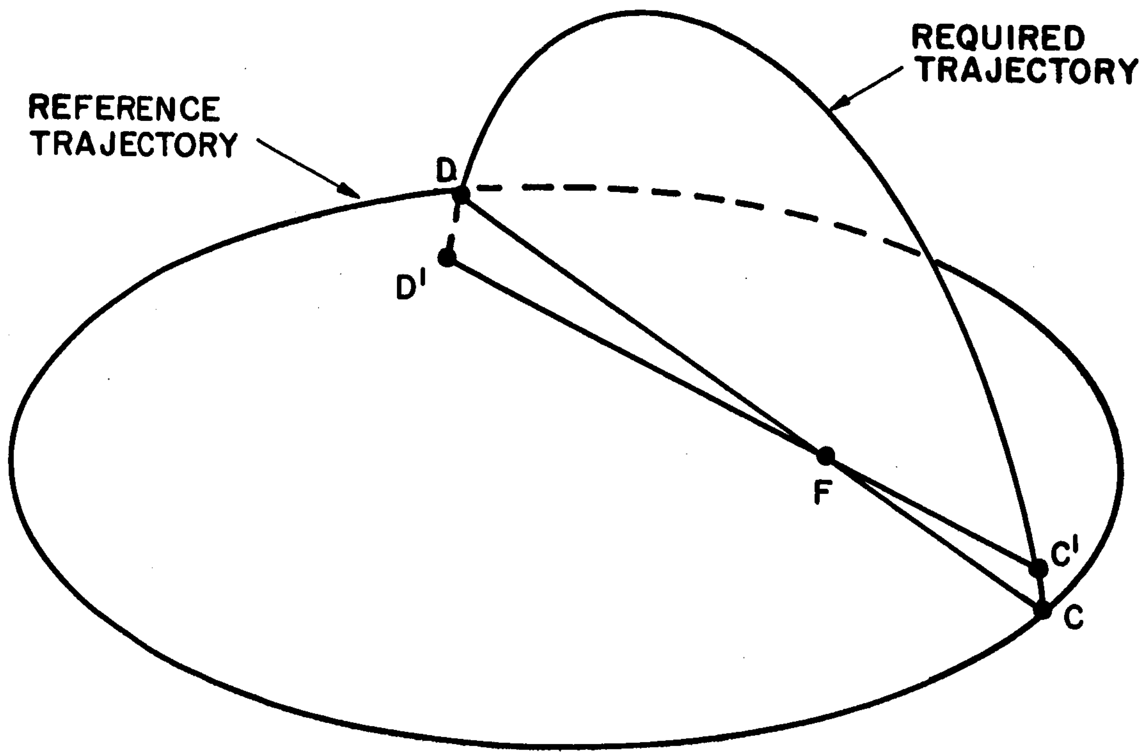
$$\delta z_j = -\frac{r_j}{r_i} \delta z_i \qquad \text{(O-23)}$$

Obviously, δz_i and δz_j cannot be used to obtain the two elements of $\delta \underline{e}$ which describe the variant motion normal to the reference trajectory plane.

If the correction time and the arrival time are such that $(f_D - f_C) = (2N - 1)\pi$, Eqs. (K-48) and (L-1) indicate that a finite FTA correction can be computed only if $\delta z_D^- = 0$. In that special case, the velocity correction vector $\underline{c}_F$ lies in the reference trajectory plane.

O.7 Physical Interpretation of Singularities at $(f_j - f_i) = (2N - 1)\pi$

Consider a body in an elliptical orbit about an attractive focus F, as shown in Fig. O.2. The orbit must lie in one plane; therefore, if the body passes through point C, it must eventually pass through some point such as D, which lies on the extension of CF through F. This statement has general validity, irrespective of the nature of the elliptical trajectory and of the inclination of the trajectory plane. Thus, the position of the body at D is not completely independent of the position at C.



F – attractive focus (sun)

D – destination point

C – position on reference trajectory corresponding to $t = t_C$

C' – position on actual trajectory corresponding to $t = t_C$

D' – predicted position at $t = t_D$ if no correction is applied

t_C – time of correction

t_D – time of arrival at destination

Figure O.2 Effect of z-Component of Position Variation when $f_D - f_C = (2N-1)\pi$

Suppose that at time t_C a vehicle is at point C' and the corresponding point on the reference trajectory is C. The distance CC' is parallel to the z-axis and small. A correction is to be applied at C' such that the vehicle will arrive at the prescribed destination point D at time t_D. If no correction is made, the vehicle's position at t_D will be D'. This example is similar to the second special case cited at the end of Section O. 5.

The corrected trajectory must contain the line segments F C' and F D. The plane containing these two segments is the r_C - z plane, which is perpendicular to the reference trajectory plane. As stated in Section O. 5, the magnitude of the velocity correction required to rotate the trajectory plane through approximately 90° is beyond the scope of linear theory and hence cannot be computed by the use of that theory.

Any small velocity correction applied in the z-direction when the vehicle is at C' has the effect of rotating the trajectory plane about the axis D' F C'. The size and shape of the orbit are not affected, so that the vehicle must pass through D' at time t_D.

O. 8 Numerical Example of Singularities at X = 0

The singularity factor X is defined by Eqs. (K-14), (K-15), and (K-17), which are repeated here for convenience.

$$X = (3\,E_M - e \sin E_M \cos E_P)(\cos E_M + e \cos E_P) - 4 \sin E_M \tag{O-24}$$

where

$$E_P = \frac{1}{2}(E_j + E_i) \tag{O-25}$$

$$E_M = \frac{1}{2}(E_j - E_i) \tag{O-26}$$

Unlike the first two types of singularities, those for which X = 0 depend on the reference trajectory, as indicated by the presence of the eccentricity e in Eq. (O-24).

Because the formulation for X in Eq. (O-24) may be considered somewhat formidable, a graph of X versus $(E_j - E_i)$ is presented in Fig. O. 3. The plot is made for a varying E_i, with e and E_j held constant. The value of e is 0. 25, a typical value for journeys to Venus or Mars. The angle E_j is 210°, which is representative of an inbound journey from Venus to Earth or an outbound journey from Earth to Mars. The plot covers the range 0° to 1800° in $(E_j - E_i)$.

Although Fig. O. 3 is drawn for specific values of e and E_j, it is characteristic of the relationship between X and $(E_j - E_i)$ for any value of e in the elliptical range and any angle E_j.

There are several interesting characteristics of the curve. It has the general appearance of a sinusoid whose amplitude is steadily increasing as $(E_j - E_i)$ gets larger. At $(E_j - E_i) = 0$, both X and its partial derivative with respect to $(E_j - E_i)$ are equal to zero.

The zero crossings of the curve are of particular interest, since those are the points at which the matrix $\overset{*}{A}_{ij}$ becomes singular. There is no zero crossing for $0° < (E_j - E_i) < 360°$. For each succeeding interval of 360° there is one zero crossing.

As $(E_j - E_i)$ gets large, the curve is dominated by the term $3\,E_M (\cos E_M + e \cos E_P)$. In the limit as the anomaly difference approaches infinity,

$$\lim_{E_M \to \infty} \left(\frac{X}{E_M}\right) = 3\,(\cos E_M + e \cos E_P)$$

$$= \frac{3}{2 \sin E_M}\,[\sin (E_j - E_i) + e\,(\sin E_j - \sin E_i)] \qquad \text{(O-27)}$$

The special case of $\sin E_M = 0$ constitutes a singularity of the first type, which has already been discussed, and hence will be ignored in this analysis.

For large values of $(E_j - E_i)$, the singularity occurs when the limit expression of Eq. (O-27) is equal to zero.

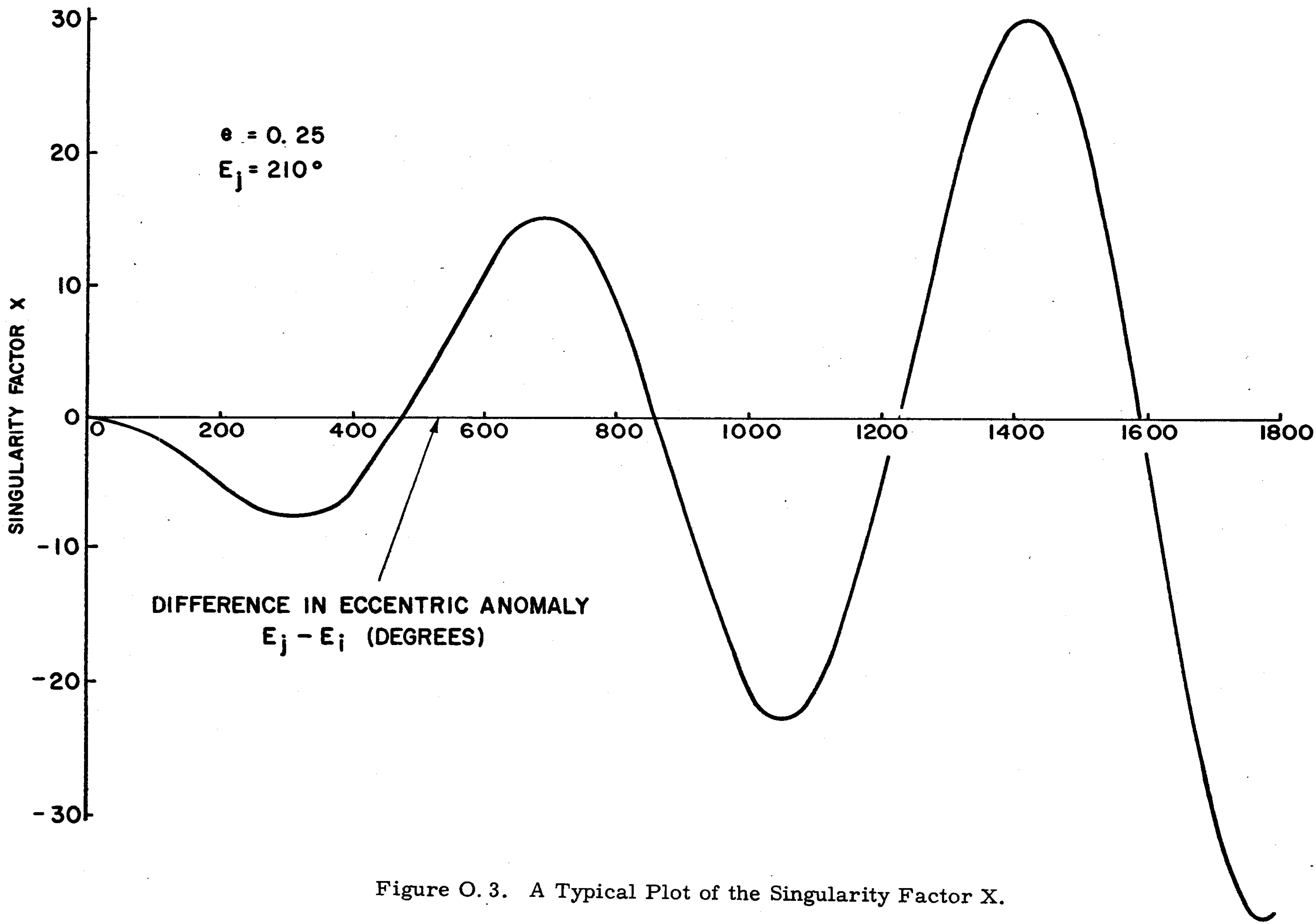


Figure O.3. A Typical Plot of the Singularity Factor X.

$$0 = \sin (E_j - E_i) + e (\sin E_j - \sin E_i)$$

$$= \sin E_j (\cos E_i + e) - \sin E_i (\cos E_j + e)$$

$$= \frac{1}{a^2 (1 - e^2)^{1/2}} [(x_i + 2 a e) y_j - (x_j + 2 a e) y_i] \qquad \text{(O-28)}$$

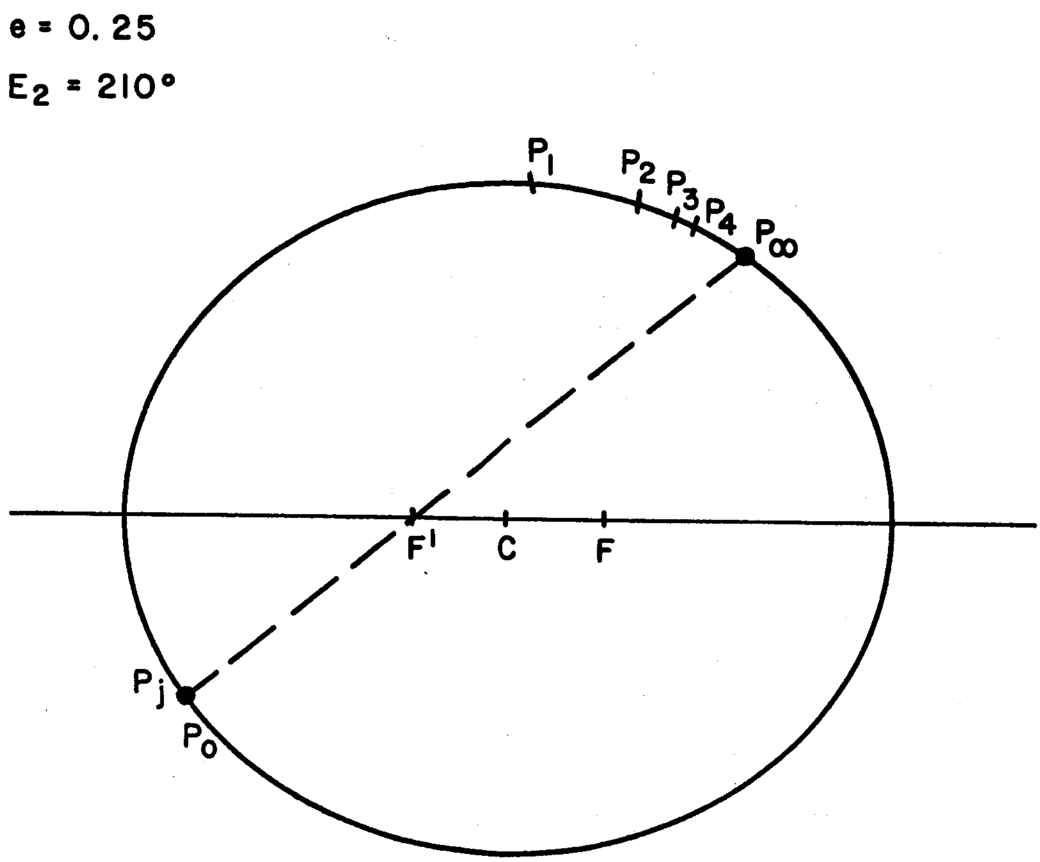
Then the condition for the existence of the singularity at large values of $(E_j - E_i)$ is

$$\frac{y_j}{x_j + 2 a e} = \frac{y_i}{x_i + 2 a e} \qquad \text{(O-29)}$$

The distances $(x_i + 2 a e)$ and y_i are, respectively, the x and y components of the distance of the point P_i on the ellipse from the vacant focus. Thus, when $(E_j - E_i)$ is very large, the singularity condition occurs when the straight line through the points P_i and P_j on the ellipse passes through the vacant focus.

Table O-1 lists the points at which the first few singularities occur, as well as the singular point for $(E_j - E_i) \to \infty$. The symbol N in the table denotes the number of complete circuits between E_i and E_j.

For a fixed point P_j, the effect of increasing N is to move the corresponding singular point P_i along the ellipse from P_j toward the point at which the straight line through P_j and the vacant focus intersects the ellipse. The singular point approaches the latter point asymptotically as N tends toward infinity. The progression of P_i is illustrated in Fig. O. 4.



C – center of ellipse

F – attractive focus

F' – vacant focus

P_j – fixed destination point on ellipse

P_o, P_1, ... P_N . . . . P_∞ – singularity points for each value of N

N – number of complete circuits between P_N and P_j

Figure O.4 Positions of the Singularities at X = 0

TABLE O-1

The Singularity Points X = 0 for e = 0.25 and E_j = 210°

N	$E_j - E_i$	$E_j - E_i - N360°$	$E_i + N360°$	$f_i + N360°$
0	0°	0°	210°	204°
1	482°	122°	88°	102°
2	860°	140°	70°	84°
3	1227°	147°	63°	77°
4	1589°	149°	61°	74°
∞	∞	162°	48°	60°

O.9 Mathematical Study of Singularities at X = 0

The singularities at X = 0 reduce the rank of the 4-by-4 sub-matrix of A^*_{ij} to three and have no effect on the rank of the 2-by-2 sub-matrix; the rank of A^*_{ij} is reduced from six to five. (This discussion of the singularities for which X = 0 does not apply to the trivial case, $E_j - E_i = 0$.)

When X = 0, there must be a linear relationship between δp_i, δq_i, δp_j, and δq_j. After some algebraic manipulation of Eq. (K-13) and the use of several of the celestial mechanics relations of Appendix B, this linear relationship may be written as

$$p_j \, \delta q_j - p_i \, \delta q_i$$

$$= -\frac{b}{2} (3 E_M - e \sin E_M \cos E_P) (\cos \gamma_j \, \delta p_j + \cos \gamma_i \, \delta p_i) \tag{O-30}$$

where $b = a(1 - e^2)^{1/2}$ is the semi-minor axis of the reference ellipse.

When correction time t_C and arrival time t_D are such that X = 0, it is possible to compute a finite FTA velocity correction only if δp_C and δq_C are related to each other in the manner defined by setting δp_j and δq_j equal to zero in Eq. (O-30) and substituting δp_C for δp_i, δq_C for δq_i. The relation between δp_C and δq_C is then given by

$$\frac{\delta p_C}{\delta q_C} = \frac{2 r_C}{b (3 E_M - e \sin E_M \cos E_P)} \tag{O-31}$$

Eq. (O-31) specifies the ratio of the two components of position variation at $t = t_C$ but does not stipulate any particular value for either. Thus, when $X = 0$, a finite FTA correction can be applied only if the position variation component in the reference trajectory plane at time t_C lies along the line defined by Eq. (O-31). In Fig. O. 5 this line is indicated as ACB.

Let μ_C be the angle between line ACB and the q_C-axis. Then,

$$\tan \mu_C = \frac{\delta p_C}{\delta q_C} \tag{O-32}$$

Because r_C and b are normally of the same order of magnitude and E_M is large at the singularity points, the angle μ_C is small. As N gets larger, μ_C gets smaller, until finally ACB is parallel to the q_C-axis when N approaches infinity. Table O-2 lists μ_C as a function of N for the conditions used in the plot of Fig. O. 3.

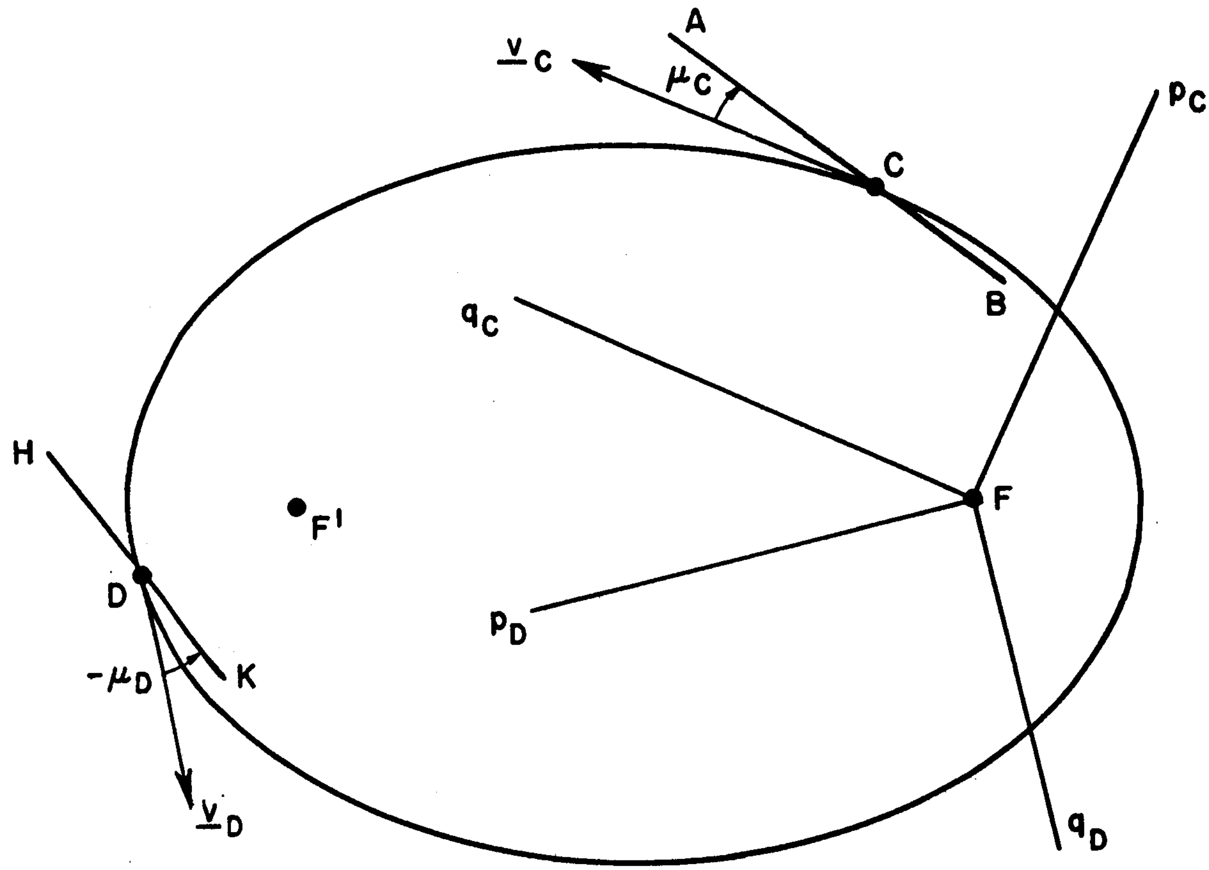
By substituting Eq. (O-31) into (O-30), a relation is obtained for the ratio of the predicted position variation components δp_D^- and δq_D^- for the special case when a finite correction can be computed at $X = 0$.

$$\frac{\delta p_D^-}{\delta q_D^-} = - \frac{2 r_D}{b (3 E_M - e \sin E_M \cos E_P)} \tag{O-33}$$

The line defined by Eq. (O-33) is HDK in Fig. O. 5. The angle between HDK and the q_D-axis is designated μ_D.

$$\tan \mu_D = \frac{\delta p_D^-}{\delta q_D^-} \tag{O-34}$$

Table O-2 lists values of μ_D as well as μ_C.



D – nominal destination point

C – singularity point corresponding to D

$\underline{v}_C$ – nominal velocity vector at C

Fp_C, Fq_C – instantaneous position of flight path system coordinate axes at $t = t_C$

ACB – straight-line locus of points at which a finite velocity correction can be computed

μ_C – angle between ACB and q_C-axis

$\underline{v}_D$ – nominal velocity vector at D

Fp_D, Fq_D – instantaneous position of flight path system coordinate axes at $t = t_D$

HDK – straight-line locus of predicted destination points if no correction is applied at $t = t_C$

μ_D – angle between HDK and q_D-axis

Figure O.5 Special Case for which Velocity Correction Can Be Computed at X = 0

Angles μ_D and μ_C are related by the equation

$$\frac{\tan \mu_D}{\tan \mu_C} = - \frac{r_D}{r_C} \qquad \text{(O-35)}$$

TABLE O-2

Angles μ_C and μ_D at X = 0 Singularity Points

e = 0.25 $\qquad$ $E_D = 210°$

N.	$\frac{\delta p_C}{\delta q_C}$	μ_C	$\frac{\delta p_D}{\delta q_D}$	μ_D
1	0.1600	9°	-0.1967	$-11\frac{1}{4}°$
2	0.0833	$4\frac{3}{4}°$	-0.1110	$-6\frac{1}{4}°$
3	0.0569	$3\frac{1}{4}°$	-0.0780	$-4\frac{1}{2}°$
4	0.0437	$2\frac{1}{2}°$	-0.0602	$-3\frac{1}{2}°$
∞	0	0°	0	0°

The velocity correction for the special case is determined by substituting Eqs. (O-33) and (K-48) into (L.1); the resulting equation is (O-36). The likelihood of the occurrence of a situation in which the correction given by Eq. (O-36) can be utilized is minuscule. The special case is of interest only in the academic sense.

O.10 Lambert's Theorem

As an introduction to the physical interpretation of the X = 0 singularities, this section presents a brief summary of Lambert's theorem and some of its ramifications. The derivation given is based on that of Plummer(29); the ensuing discussion is related to the work of Battin(1).

$$\underline{c}_F = -\frac{n}{2}\left\{\begin{array}{c} \dfrac{1}{(1 - e^2 \cos^2 E_C)^{1/2} (1 - e^2 \cos^2 E_D)^{1/2}} \begin{pmatrix} \dfrac{\cos E_M + e \cos E_P}{\sin E_M} & 0 \\ \dfrac{1}{2}(1 - e^2)^{1/2}(1 - e \cos E_C) & 0 \end{pmatrix} \\ \hline \begin{pmatrix} 0 & \dfrac{1}{(\cos E_M - e \cos E_P) \sin E_M} \end{pmatrix} \end{array}\right\} \begin{Bmatrix} \delta p_D^- \\ \delta z_D^- \end{Bmatrix} \quad \text{(O-36)}$$

The Lambert problem may be stated as follows: A body is moving about an attractive focus at F in an elliptical trajectory whose semi-major axis is a. It is desired to find an expression for the time required by the body to travel from an arbitrary point P to an arbitrary point Q on the trajectory. This expression is to be independent of the eccentricity of the trajectory.

The given data consist of the space triangle FPQ in Fig. O.6 and the semi-major axis length a. The position of F', the vacant focus, is not known; neither is the eccentricity e. The known distances FP, FQ, and PQ are designated r_1, r_2, and d, respectively.

The subscript 1 is used for conditions at point P; the subscript 2 is used for conditions at point Q. The time of flight t_F is

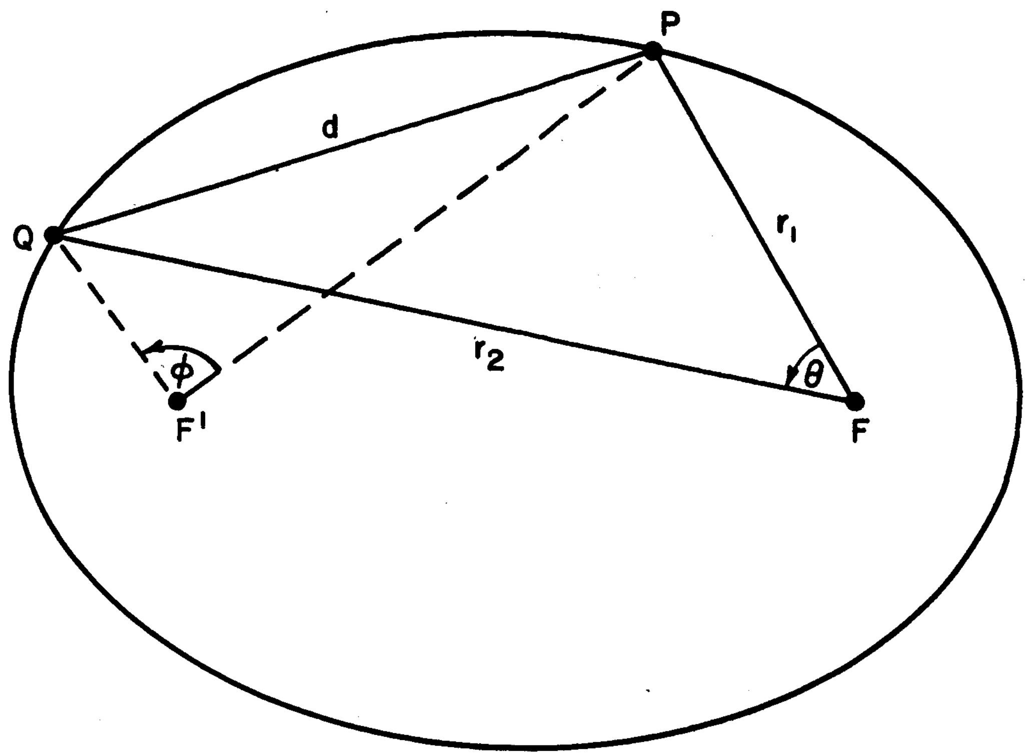
$$t_F = t_2 - t_1 \quad \text{(O-37)}$$

In this section, E_P and E_M are given by

$$E_P = \frac{1}{2}(E_2 + E_1) \quad \text{(O-38)}$$

$$E_M = \frac{1}{2}(E_2 - E_1) \quad \text{(O-39)}$$

Three additional angles are used in the derivation. They are defined by the following equations:



F – attractive focus
F' – vacant focus
P – initial position
Q – final position

Figure O.6 Illustration for Lambert's Theorem

$$\cos \eta = e \cos E_P \tag{O-40}$$

$$\alpha = \eta + E_M \tag{O-41}$$

$$\beta = \eta - E_M \tag{O-42}$$

With the aid of Eqs. (B-45), (B-55), and (B-62), the time-of-flight equation may be written as

$$t_F = \frac{1}{n}(M_2 - M_1)$$

$$= \frac{1}{n}[(E_2 - E_1) - e(\sin E_2 - \sin E_1)]$$

$$= \frac{2}{n}(E_M - \cos \eta \sin E_M)$$

$$= \left(\frac{a^3}{\mu}\right)^{1/2} [(\alpha - \beta) - (\sin \alpha - \sin \beta)] \tag{O-43}$$

Eq. (O-43) in itself does not solve the Lambert problem, since α and β are known only in terms of e. The task now is to express α and β in terms of the known quantities r_1, r_2, d, and a. As a start, $(r_1 + r_2)$ and d are found in terms of a, η, and E_M.

$$r_1 + r_2 = a(1 - e \cos E_1) + a(1 - e \cos E_2)$$

$$= 2a(1 - \cos \eta \cos E_M) \tag{O-44}$$

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$= a^2 (\cos E_2 - \cos E_1)^2 + a^2 (1 - e^2)(\sin E_2 - \sin E_1)^2$$

$$= 4 a^2 \sin^2 \eta \sin^2 E_M \tag{O-45}$$

$$d = 2a \sin \eta \sin E_M \tag{O-46}$$

By first adding Eq. (O. 46) to (O-44) and then subtracting Eq. (O-46) from (O-44), it may be shown that

$$\sin^2 \frac{\alpha}{2} = \frac{r_1 + r_2 + d}{4\,a} \tag{O-47}$$

$$\sin^2 \frac{\beta}{2} = \frac{r_1 + r_2 - d}{4\,a} \tag{O-48}$$

The combination of Eqs. (O-43), (O-47), and (O-48) constitutes the solution of the Lambert problem. By means of the three equations, t_F is determined as a function of $(r_1 + r_2)$, d, and a.

There are two possible sources of ambiguity when Eqs. (O-47) and (O-48) are used to compute α and β. The first arises from the sign of the square root; the second involves the determination of the quadrant of an angle whose sine is known. The ambiguities may be resolved by arbitrary definitions in Eqs. (O-47) and (O-48), and modification of Eq. (O-43) to accommodate these definitions.

The positive sign is chosen for $\sin(\alpha/2)$ and $\sin(\beta/2)$. It is further stipulated that both $\alpha/2$ and $\beta/2$ lie in the first quadrant. Then the following inequality defines the ranges of α and β:

$$0 \leq \beta \leq \alpha \leq \pi \tag{O-49}$$

Eq. (O-43) must be revised not only because of the arbitrary definitions of α and β but also due to the fact that there may be N complete circuits of the focus between t_1 and t_2. The revised equation is

$$t_F = \left(\frac{a^3}{\mu}\right)^{1/2} [(2\,N + 1)\pi + \mathrm{sgn}\,(\sin\phi)\,(\alpha - \sin\alpha - \pi) - \mathrm{sgn}\,(\sin\theta)\,(\beta - \sin\beta)] \tag{O-50}$$

The angles θ and ϕ are shown in Fig. O. 6. θ is the angle subtended at F by the initial position P and the final position Q; ϕ is the angle subtended at F' by P and Q. Both θ and ϕ are positive in the direction of the orbital motion. The symbol "sgn", or signum, is defined by the relations

$$\begin{aligned} \operatorname{sgn}(x) &= +1 \quad \text{if} \quad x > 0 \\ \operatorname{sgn}(x) &= 0 \quad \text{if} \quad x = 0 \\ \operatorname{sgn}(x) &= -1 \quad \text{if} \quad x < 0 \end{aligned} \tag{O-51}$$

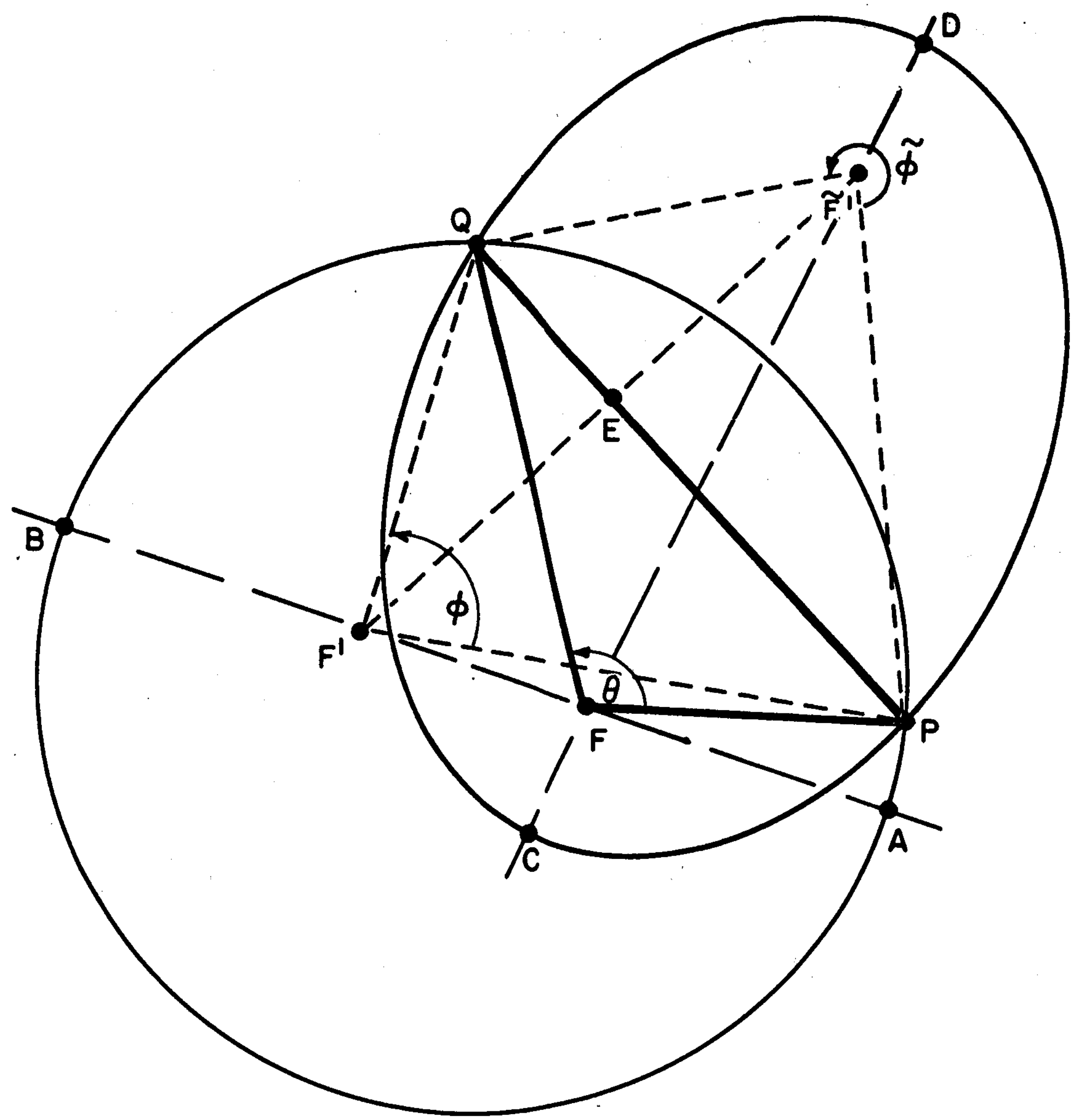
Eq. (O-50) can be solved for the proper time of flight for any combination of P, Q, and F except for the case when points P and Q coincide. In this special case, the signum notation causes an incorrect result; the correct time of flight is simply N times the period.

There is also a restriction on the semi-major axis; it must be large enough so that the values of $\sin^2(\alpha/2)$ and $\sin^2(\beta/2)$, obtained from Eqs. (O-47) and (O-48), are never larger than unity. Thus, the minimum value of a is

$$a_{min} = \frac{r_1 + r_2 + d}{4} \tag{O-52}$$

Battin(1) has shown that for a given space triangle FPQ and a given value of a which is greater than a_{min}, there are two possible elliptical paths from P to Q. These are shown in Fig. O. 7. The two vacant focus positions are F' and $\tilde{F}'$. The line PQ is the perpendicular bisector of the line joining F' and $\tilde{F}'$.

Since the value of a is the same for the two ellipses, they both have the same period. Kepler's second law states that the radius vector (from F) sweeps through equal areas in equal times. Then, for each of the two ellipses the time of flight from P to Q is equal to the period times the ratio, for that ellipse, of the area of the sector FPQF to the total area of the ellipse. If all motion is assumed to be counter-clockwise in Fig. O. 7, it is apparent that the area ratio for the ellipse whose vacant focus is F' is less than the area ratio for the ellipse with vacant focus at $\tilde{F}'$, and therefore, t_F, the time of flight for the former ellipse, is less than $\tilde{t}_F$, the time of flight for the latter.



F – attractive focus
P – initial position
Q – final position
$F', \tilde{F}'$ – two permissible vacant foci for same value of a
$AB = CD = 2a$ = length of major axis of each ellipse
$F'\tilde{F}' \perp PQ$; $F'E = E\tilde{F}'$
$\phi + \tilde{\phi} = 2\pi$

Figure O.7 The Two Ellipses for a Given Space Triangle and a Given Length of the Major Axis

The relation between t_F and $\tilde{t}_F$ can be developed mathematically from Eq. (O-50). With the exception of the angle ϕ, all the quantities on the right-hand side of that equation are the same for both ellipses of Fig. O.7. The coefficient of sgn (sin ϕ) in the equation is $(\alpha - \sin \alpha - \pi)$ which, for $a > a_{min}$, is always negative. Therefore, the time of flight for a value of ϕ less than π radians is smaller than the time of flight corresponding to ϕ greater than π radians. Because the triangles PF'Q and P $\tilde{F}$' Q are congruent, the sum of angles ϕ and $\tilde{\phi}$, shown in the figure, is 2π radians. Since the angles are not equal except in the special case $a = a_{min}$, one must be less than π radians and the other greater than π radians. If the tilde notation is associated with the value of ϕ greater than π, $\tilde{t}_F$ is always greater than t_F. The difference between $\tilde{t}_F$ and t_F is

$$\tilde{t}_F - t_F = 2 \left(\frac{a^3}{\mu}\right)^{1/2} (\pi - \alpha + \sin \alpha) \qquad \text{(O-53)}$$

O.11 Minimum Time of Flight

In this section it will be shown that, for a given space triangle FPQ, the rate of change of the time of flight with change in semi-major axis is proportional to the factor X, and consequently a singularity of the X = 0 type occurs when the time of flight is a minimum.

Figure O.8 is a plot of t_F vs. a for a journey from Earth to Mars: For such a journey $r_1 = 1$ astronomical unit (a.u.), and $r_2 = 1.524$ a.u. Curves are presented for three values of θ at $N = 0$ and for the same three values of θ at $N = 1$. Figure O.8 duplicates the curves of Fig. 3-3 of Reference (1).

It may be noted that the curves corresponding to $N = 0$ have no minimum values of t_F for any finite value of a. As a is increased beyond a_{min}, the two possible values of t_F get farther and farther apart, one continuously increasing and the other continuously decreasing.

Each of the curves for $N = 1$ has a definite minimum value of t_F; the minimum t_F for each curve occurs at a value of a that is slightly larger than a_{min} for that curve.

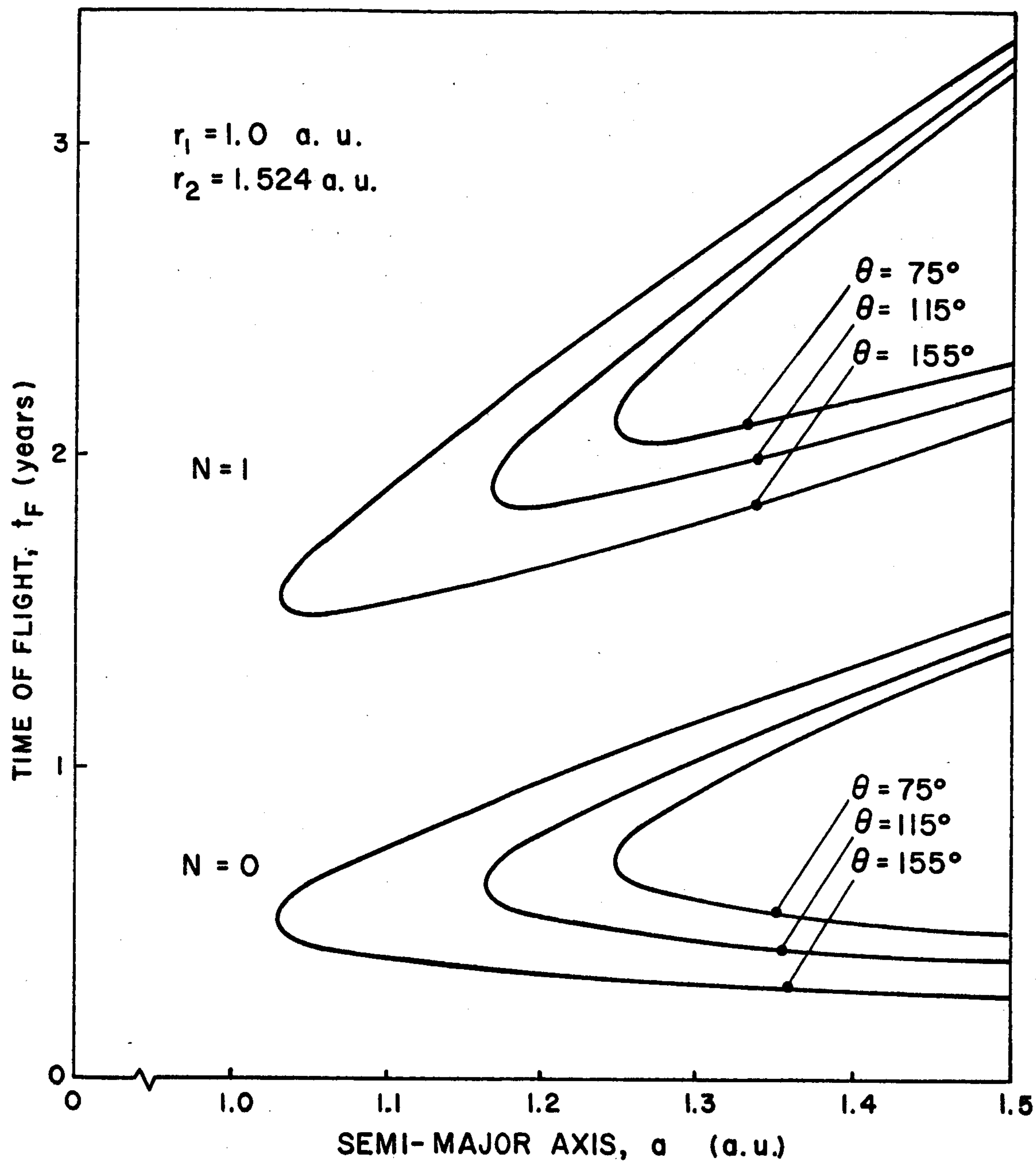


Figure O.8 Time of Flight for One-Way Trip from Earth to Mars

If curves were drawn for values of N greater than one, they would also exhibit the characteristic of a minimum t_F. As N gets larger, for a given θ, the distance between a_{min} and the a corresponding to $t_{F_{min}}$ gets smaller.

In order to gain further insight into the time-of-flight curves and their minima, an analytic expression for the slope will be derived. In this section the notation $\partial(\ )/\partial a$ signifies the partial derivative of the argument with respect to a, with r_1, r_2, d, and N all constant. Inasmuch as there are no ambiguities involved in the differentiation, the expression for t_F given by Eq. (O-43) will be used rather than the more complicated Eq. (O-50).

$$\frac{\partial t_F}{\partial a} = \frac{1}{2}\left(\frac{a}{\mu}\right)^{1/2}\left[3(\alpha - \beta) - 3(\sin\alpha - \sin\beta) + 2a(1-\cos\alpha)\frac{\partial\alpha}{\partial a} - 2a(1-\cos\beta)\frac{\partial\beta}{\partial a}\right] \quad \text{(O-54)}$$

$\partial\alpha/\partial a$ is obtained by differentiating Eq. (O-47).

$$\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\frac{\partial\alpha}{\partial a} = -\frac{r_1 + r_2 + d}{4a^2} \quad \text{(O-55)}$$

$$\frac{\partial\alpha}{\partial a} = -\frac{1-\cos\alpha}{a\sin\alpha} = -\frac{1}{a}\tan\frac{\alpha}{2} \quad \text{(O-56)}$$

$\partial\beta/\partial a$ is obtained in similar fashion from Eq. (O-48)

$$\frac{\partial\beta}{\partial a} = -\frac{1-\cos\beta}{a\sin\beta} = -\frac{1}{a}\tan\frac{\beta}{2} \quad \text{(O-57)}$$

Eqs. (O-56) and (O-57) are substituted into (O-54). After some trigonometric manipulation, the result is

$$\frac{\partial t_F}{\partial a} = \frac{1}{2}\left(\frac{a}{\mu}\right)^{1/2}\left[3(\alpha - \beta) - (\sin\alpha - \sin\beta) - 4\left(\tan\frac{\alpha}{2} - \tan\frac{\beta}{2}\right)\right] \tag{O-58}$$

The tangents of the half-angles in Eq. (O-58) may be replaced in the following way:

$$\begin{aligned}
&-4\left(\tan\frac{\alpha}{2} - \tan\frac{\beta}{2}\right) \\
&= -4\left(\frac{1-\cos\alpha}{\sin\alpha} - \frac{1-\cos\beta}{\sin\beta}\right) \\
&= -\frac{4\,[\sin(\alpha-\beta) - (\sin\alpha - \sin\beta)]}{\sin\alpha\,\sin\beta} \\
&= -\frac{4\,[\sin^2(\alpha-\beta) - (\sin\alpha - \sin\beta)^2]}{[\sin(\alpha-\beta) + (\sin\alpha - \sin\beta)]\,\sin\alpha\,\sin\beta} \\
&= -\frac{8\,[1-\cos(\alpha-\beta)]}{\sin(\alpha-\beta) + (\sin\alpha - \sin\beta)}
\end{aligned} \tag{O-59}$$

Eqs. (O-58) and (O-59) are combined.

$$\frac{\partial t_F}{\partial a} = \frac{\frac{1}{2}\left(\frac{a}{\mu}\right)^{1/2}}{\sin(\alpha-\beta) + (\sin\alpha - \sin\beta)} \cdot \Big\{[3(\alpha-\beta) - (\sin\alpha - \sin\beta)]\,[\sin(\alpha-\beta) + (\sin\alpha - \sin\beta)] - 8\,[1-\cos(\alpha-\beta)]\Big\} \tag{O-60}$$

It will now be shown that the quantity within the braces in Eq. (O-60) is proportional to the singularity factor X.

From Eqs. (O-40), (O-41), and (O-42),

$$\alpha - \beta = 2\,E_M \tag{O-61}$$

$$\alpha + \beta = 2\,\eta \tag{O-62}$$

$$\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\,\sin\frac{\alpha-\beta}{2}$$

$$= 2\cos\eta\,\sin E_M$$

$$= 2e\cos E_P\,\sin E_M \tag{O-63}$$

Eqs. (O-61) and (O-63) are substituted into (O-60), and both numerator and denominator are divided by 2 sin E_M.

$$\frac{\partial t_F}{\partial a} = \frac{\left(\frac{a}{\mu}\right)^{1/2}}{\cos E_M + e\cos E_P}$$

$$\cdot\ [(3\,E_M - e\sin E_M\cos E_P)(\cos E_M + e\cos E_P) - 4\sin E_M] \tag{O-64}$$

The quantity inside the brackets is identical with the expression for X in Eq. (O-24). Finally,

$$\frac{\partial t_F}{\partial a} = \frac{\left(\frac{a}{\mu}\right)^{1/2} X}{\cos E_M + e\cos E_P} \tag{O-65}$$

Thus, it has been proved that the slope of each time-of-flight curve is proportional to X, and the minimum time of flight, if it exists for a particular curve, occurs at the X = 0 singularity point for that curve.

O. 12 Physical Interpretation of Singularities at X = 0

In the preceding section it was shown that the time of flight, as depicted in the curves of Fig. O. 8, is insensitive to small changes in the semi-major axis when X = 0. In this section, it will be proved that, for a given vector $\underline{r}_1$ and given values of t_F and N, the vector $\underline{r}_2$ is insensitive to small changes in the semi-major axis when X = 0, and consequently it is not possible to apply a small velocity correction at t_1 which will alter $\underline{r}_2$.

Consider a case in which r_1, t_F, d, and N are specified, and r_2 is regarded as a function of a. The partial derivative of the time-of-flight equation, (O-43), is taken with respect to a.

$$0 = \frac{3}{2}\left(\frac{a}{\mu}\right)^{1/2}[(\alpha - \beta) - (\sin\alpha - \sin\beta)]$$
$$+\left(\frac{a^3}{\mu}\right)^{1/2}\left[(1-\cos\alpha)\left(\frac{\partial\alpha}{\partial a}\right)_{r_1,\, t_F,\, d,\, N} - (1-\cos\beta)\left(\frac{\partial\beta}{\partial a}\right)_{r_1,\, t_F,\, d,\, N}\right] \qquad \text{(O-66)}$$

The subscript symbols following the partial derivative indicate the quantities that are being held constant.

From Eqs. (O-47) and (O-48), the partial derivatives in Eq. (O-66) may be expressed as

$$\left(\frac{\partial\alpha}{\partial a}\right)_{r_1,\, t_F,\, d,\, N} = \frac{1}{2a\sin\alpha}\left[\left(\frac{\partial r_2}{\partial a}\right)_{r_1,\, t_F,\, d,\, N} - 4\sin^2\frac{\alpha}{2}\right] \qquad \text{(O-67)}$$

$$\left(\frac{\partial \beta}{\partial a}\right)_{r_1,\ t_F,\ d,\ N} = \frac{1}{2\,a\,\sin\beta}\left[\left(\frac{\partial r_2}{\partial a}\right)_{r_1,\ t_F,\ d,\ N} - 4\sin^2\frac{\beta}{2}\right] \tag{O-68}$$

Eqs. (O-66), (O-67), and (O-68) are combined and solved for $(\partial r_2/\partial a)_{r_1,\ t_F,\ d,\ N}$.

$$\left(\frac{\partial r_2}{\partial a}\right)_{r_1,\ t_F,\ d,\ N} = -\frac{X}{\sin E_M} \tag{O-69}$$

The derivation of Eq. (O-69) involves a division of numerator and denominator by $\sin E_M$; therefore, the equation is not valid for the special case when $\sin E_M = 0$; i.e., when $(E_2 - E_1) = 2N\pi = (f_2 - f_1)$. In general, for $\sin E_M \neq 0$, the rate of change of r_2 with a is proportional to X.

As a second case, consider r_1, t_F, r_2, and N to be specified, and d to be a function of a. The same procedure as that of the first example is followed. The equations analogous to Eqs. (O-66) through (O-69) are

$$\begin{aligned} 0 = {} & \frac{3}{2}\left(\frac{a}{\mu}\right)^{1/2}[(\alpha - \beta) - (\sin\alpha - \sin\beta)] \\ & + \left(\frac{a^3}{\mu}\right)^{1/2}\left[(1-\cos\alpha)\left(\frac{\partial\alpha}{\partial a}\right)_{r_1,\ t_F,\ r_2,\ N}\right. \\ & \left. - (1-\cos\beta)\left(\frac{\partial\beta}{\partial a}\right)_{r_1,\ t_F,\ r_2,\ N}\right] \end{aligned} \tag{O-70}$$

$$\left(\frac{\partial \alpha}{\partial a}\right)_{r_1,\ t_F,\ r_2,\ N} = \frac{1}{2a\sin\alpha}\left[\left(\frac{\partial d}{\partial a}\right)_{r_1,\ t_F,\ r_2,\ N} - 4\sin^2\frac{\alpha}{2}\right] \tag{O-71}$$

$$\left(\frac{\partial\beta}{\partial a}\right)_{r_1,\ t_F,\ r_2,\ N} = \frac{1}{2\ a\ \sin\beta}\left[\left(\frac{\partial d}{\partial a}\right)_{r_1,\ t_F,\ r_2,\ N} - 4\ \sin^2\frac{\beta}{2}\right] \tag{O-72}$$

$$\left(\frac{\partial d}{\partial a}\right)_{r_1,\ t_F,\ r_2,\ N} = -\frac{X}{\sin\eta} \tag{O-73}$$

From Eq. (O-40),

$$\sin\eta = (1 - e^2 \cos^2 E_P)^{1/2} \tag{O-74}$$

Because the denominator term in Eq. (O-73) is obtained by dividing $+\sin^2\eta$ by $+\sin\eta$, the positive sign must be used for the root in Eq. (O-74). For values of e less than one, $\sin\eta$ cannot be zero, and hence Eq. (O-73) always produces a finite value of the partial derivative.

Eqs. (O-69) and (O-73) indicate that $(\partial r_2/\partial a)_{r_1,\ t_F,\ d,\ N}$ and $(\partial d/\partial a)_{r_1,\ t_F,\ r_2,\ N}$ are proportional to X. For given values of r_1, t_F, and N, if X = 0, the distances r_2 and d are unaffected by a small change in the length of the semi-major axis. From Figure O.6 it is apparent that, if F and P are fixed points and r_2 and d are unaffected by small changes in a, then the point Q is unaffected by small changes in a. Therefore, under the given conditions, the vector $\underline{r}_2$ is insensitive to small changes in a when X = 0. Vectors $\underline{r}_1$ and $\underline{r}_2$ are not independent when X = 0.

When small changes in the semi-major axis a are mentioned, it should not be inferred that the other three orbital elements defining motion in the trajectory plane are unaffected. The changes in all four elements must be related in such a manner that the given vector $\underline{r}_1$ is conserved. Then the conclusion reached in the preceding paragraph may be generalized to indicate that, with $\underline{r}_1$, t_F, and N specified such that X = 0, the position vector $\underline{r}_2$ is not affected by small changes in the orbital elements defining motion in the trajectory plane.

Therefore, it is not possible to compute a small step change in velocity which, if applied at t_1, will alter (i.e., "correct") the position of the vehicle at t_2.

In recapitulation, the developments in the last three sections establish a connection between the X = 0 singularities, which evolve from linear perturbation theory, and Lambert's theorem in celestial mechanics. Whenever there is a minimum in the curve of time of flight versus semi-major axis length for fixed values of r_1, r_2, d, and N, that minimum occurs under conditions for which X = 0. By the use of partial differentiation on the time-of-flight equation of Lambert, it has been shown that, for a given $\underline{r}_1$, the position vector $\underline{r}_2$ is unaffected by small changes in the orbital elements when X = 0, and consequently it is not possible to compute a small velocity correction which, if applied at t_1, changes the vehicle's predicted position at t_2.

The special case discussed in Section O.9, for which it is possible to compute a velocity correction even though X = 0, is not explained by the analysis that has been presented in this section.

O.13 Analytic Formulation of the VTA Velocity Correction

It has already been shown that, in general, it is not possible to compute a finite FTA velocity correction when t_C and t_D are related in such a manner that any one of the three types of singularities exists. In the following sections, the feasibility of applying a finite VTA correction under these conditions will be investigated.

In the critical-plane coordinate system the relations defining the VTA correction are Eqs. (N-27) and (N-46), which are repeated here for convenience.

$$\underline{c}_W = \overset{*}{Y} (\delta\underline{\rho}^-)_W \tag{O-75}$$

where

$$
\overset{*}{Y} = \left\{ \begin{array}{ll}
\frac{1}{A} (k_{12} k_{21} - k_{11} k_{22}) & 0 \\
\frac{1}{A} \frac{v_R}{w} k_{33} \cos i_D \, [(k_{11}{}^2 - k_{12}{}^2 & \\
+ k_{21}{}^2 - k_{22}{}^2) \sin \Omega_D \cos \Omega_D & - A \frac{v_R}{w} k_{33} \\
+ (k_{11} k_{12} + k_{21} k_{22}) (\sin^2 \Omega_D & \\
- \cos^2 \Omega_D)] &
\end{array} \right\}
$$

(O-76)

In order to emphasize the effect of the singularities, the elements k_{11}, k_{12}, k_{21}, k_{22}, and k_{33} will be replaced by K_{11}, K_{12}, K_{21}, K_{22}, and K_{33}, respectively. The new terms are defined in such a manner that they remain finite even under the singularity conditions. With the aid of Eq. (K-48), they are expressed as follows:

$$
K_{11} = \frac{1}{n} X \sin E_M k_{11}
$$

$$
= \frac{(1 + e \cos E_C)(1 + e \cos E_D)(3 E_M - e \sin E_M \cos E_P) - 4 \sin E_M (\cos E_M + e \cos E_P)}{2 (1 - e^2 \cos^2 E_C)^{1/2} (1 - e^2 \cos^2 E_D)^{1/2}}
$$

(O-77)

$$
K_{12} = \frac{1}{n} X \sin E_M k_{12}
$$

$$
= \frac{(1 - e^2)^{1/2} (1 - e \cos E_D) \sin^2 E_M}{(1 - e^2 \cos^2 E_C)^{1/2} (1 - e^2 \cos^2 E_D)^{1/2}} \qquad \text{(O-78)}
$$

$$K_{21} = \frac{1}{n} X \sin E_M k_{21}$$

$$= - \frac{(1 - e^2)^{1/2} (1 - e \cos E_C) \sin^2 E_M}{(1 - e^2 \cos E_C)^{1/2} (1 - e^2 \cos^2 E_D)^{1/2}} \qquad \text{(O-79)}$$

$$K_{22} = \frac{1}{n} X \sin E_M k_{22}$$

$$= - \frac{(1 - e \cos E_C)(1 - e \cos E_D)(\cos E_M + e \cos E_P) \sin E_M}{2 (1 - e^2 \cos^2 E_C)^{1/2} (1 - e^2 \cos^2 E_D)^{1/2}} \qquad \text{(O-80)}$$

$$K_{33} = \frac{1}{n} \sin (f_D - f_C) k_{33}$$

$$= \frac{\sin (f_D - f_C)}{2 \sin E_M (\cos E_M - e \cos E_P)}$$

$$= \frac{(1 - e^2)^{1/2}}{(1 - e \cos E_C)(1 - e \cos E_D)} \qquad \text{(O-81)}$$

The terms A and v_R/w in Eq. (O-76) can be expressed as functions of the K's. From Eq. (N-42),

$$A = [(k_{11} \sin \Omega_D - k_{12} \cos \Omega_D)^2 + (k_{21} \sin \Omega_D - k_{22} \cos \Omega_D)^2]^{1/2}$$

$$= \frac{n B}{X \sin E_M} \qquad \text{(O-82)}$$

where

$$B = [(K_{11} \sin \Omega_D - K_{12} \cos \Omega_D)^2 + (K_{21} \sin \Omega_D - K_{22} \cos \Omega_D)^2]^{1/2} \tag{O-83}$$

From Eq. (N-44),

$$\frac{v_R}{w} = (A^2 \sin^2 i_D + k_{33}{}^2 \cos^2 i_D)^{-1/2}$$

$$= \frac{1}{n}\left[\frac{B^2 \sin^2 i_D}{X^2 \sin^2 E_M} + \frac{K_{33}{}^2 \cos^2 i_D}{\sin^2 (f_D - f_C)}\right]^{-1/2} \tag{O-84}$$

The parameter B, like the K factors, remains finite at the singularity points.

The elements of $\overset{*}{Y}$ can now be written in terms of the K's. A simple form for the upper left-hand term, y_{11}, is obtained by the use of Eq. (K-53).

$$y_{11} = \frac{1}{A}(k_{12} k_{21} - k_{11} k_{22})$$

$$= \frac{\dfrac{n^2}{4 X \sin E_M}}{\dfrac{n B}{X \sin E_M}} = \frac{n}{4B} \tag{O-85}$$

The expression evolved for $\overset{*}{Y}$ is

$$\overset{*}{Y} = n \left\{ \begin{array}{c} \begin{pmatrix} \dfrac{1}{4B} & 0 \end{pmatrix} \\ \hline \dfrac{K_{33}}{C} \begin{pmatrix} \dfrac{D \cos i_D}{B} & -B \end{pmatrix} \end{array} \right\} \tag{O-86}$$

where

$$C = [B^2 \sin^2 (f_D - f_C) \sin^2 i_D + K_{33}^2 X^2 \sin^2 E_M \cos^2 i_D]^{1/2} \tag{O-87}$$

$$D = (K_{11}^2 - K_{12}^2 + K_{21}^2 - K_{22}^2) \sin \Omega_D \cos \Omega_D$$

$$+ (K_{11} K_{12} + K_{21} K_{22}) (\sin^2 \Omega_D - \cos^2 \Omega_D) \tag{O-88}$$

The effect of each of the three types of singularities on the elements of $\overset{*}{Y}$ in Eq. (O-86) will now be investigated.

O. 14 Effect on VTA Guidance of Singularities at $(t_D - t_C) = NP$

When $(t_D - t_C)$ is very close to NP, both $\sin E_M$ and $\sin (f_D - f_C)$ may be equated to ϵ, a small quantity which reduces to zero when $(t_D - t_C)$ equals NP. From Eqs. (O-10), (O-11), and (O-12),

$$\cos E_M + e \cos E_P = (-1)^N (1 + e \cos E_D) \tag{O-89}$$

$$X = (-1)^N \, 3 N \pi \, (1 + e \cos E_D) \tag{O-90}$$

When ϵ is small, the K factors are given by

$$K_{11} = \frac{3 N \pi (1 + e \cos E_D)}{2 (1 - e \cos E_D)} \tag{O-91}$$

$$K_{12} = \frac{(1 - e^2)^{1/2} \epsilon^2}{1 + e \cos E_D} \tag{O-92}$$

$$K_{21} = -\frac{(1 - e^2)^{1/2} \epsilon^2}{1 + e \cos E_D} \tag{O-93}$$

$$K_{22} = \frac{1}{2} (-1)^{N+1} (1 - e \cos E_D) \epsilon \tag{O-94}$$

$$K_{33} = \frac{(1 - e^2)^{1/2}}{(1 - e \cos E_D)^2} \quad \text{(O-95)}$$

The expressions for parameters B, C, and D are

$$B = K_{11} \sin \Omega_D = \frac{3 N \pi (1 + e \cos E_D) \sin \Omega_D}{2 (1 - e \cos E_D)} \quad \text{(O-96)}$$

$$C = (B^2 \sin^2 i_D + K_{33}{}^2 X^2 \cos^2 i_D)^{1/2} \epsilon$$

$$= \left[\frac{3 N \pi (1 + e \cos E_D)}{2 (1 - e \cos E_D)^2} \right] [(1 - e \cos E_D)^2 \sin^2 \Omega_D \sin^2 i_D$$

$$+ 4 (1 - e^2) \cos^2 i_D]^{1/2} \epsilon \quad \text{(O-97)}$$

$$D = K_{11}{}^2 \sin \Omega_D \cos \Omega_D = \left[\frac{3 N \pi (1 + e \cos E_D)}{2 (1 - e \cos E_D)} \right]^2 \sin \Omega_D \cos \Omega_D$$

(O-98)

Because C approaches zero as $(t_D - t_C)$ approaches N P, the elements of the second row of $\overset{*}{Y}$ in Eq. (O-86) becomes infinite at the singularity points. Consequently, it is not possible in the general case to compute a finite VTA velocity correction when $(t_D - t_C)$ is equal to N P.

There are three special cases in which the vehicle can be made to reach the proper destination by means of VTA guidance even though a correction is contemplated at a time such that $(t_D - t_C) = NP$. The first special case is the trivial one which occurs when $\delta \underline{r}_D{}^-$ is parallel to $\underline{v}_R$. Then no correction is required, and the change in the time of arrival is proportional to the magnitude of $\delta \underline{r}_D{}^-$.

The second special case occurs when $\delta \underline{r}_D^-$ is parallel to the orbital velocity vector $\underline{v}_D$. Then

$$(\delta\rho^-)_W = \left\{ \begin{array}{c} \sin \Omega_D \\ \\ \cos \Omega_D \cos i_D \end{array} \right\} \delta q_D^- \qquad \text{(O-99)}$$

$$\underline{c}_W = \left\{ \begin{array}{c} c_\xi \\ \\ c_\eta \end{array} \right\} = n \left\{ \begin{array}{c} \left(\begin{array}{cc} \dfrac{1}{4B} & 0 \end{array} \right) \\ \hline \dfrac{K_{33}}{C} \left(\begin{array}{cc} \dfrac{D \cos i_D}{B} & -B \end{array} \right) \end{array} \right\} \cdot \left\{ \begin{array}{c} \sin \Omega_D \\ \\ \cos \Omega_D \cos i_D \end{array} \right\} \delta q_D^-$$

(O-100)

For $(t_D - t_C)$ very close to N P, the expression for c_η obtained from Eq. (O-100) is

$$c_\eta = \frac{n K_{33} \cos i_D}{B\,C} (D \sin \Omega_D - B^2 \cos \Omega_D)\, \delta q_D^- \qquad \text{(O-101)}$$

It will now be shown that c_η, which ordinarily goes to infinity at a singularity point because the demoninator factor C reduces to zero, is equal to zero in this special case. c_η can be regarded as consisting of three factors, $n K_{33} \cos i_D/B$, $1/C$, and $(D \sin \Omega_D - B^2 \cos \Omega_D)$. From Eqs. (O-95) and (O-96) it is apparent that, as long as neither $\sin \Omega_D$ nor $\cos i_D$ is zero, the first factor is non-zero and finite. Eq. (O-97) indicates that C is an infinitesimal of order ϵ. When terms of higher order than ϵ^2 are neglected, the third factor may be treated as follows:

$$(D \sin \Omega_D - B^2 \cos \Omega_D) = (K_{11}^{\ 2} - K_{22}^{\ 2}) \sin^2 \Omega_D \cos \Omega_D$$

$$+ K_{11} K_{12} (\sin^2 \Omega_D - \cos^2 \Omega_D) \sin \Omega_D$$

$$- (K_{11}^{\ 2} \sin^2 \Omega_D - 2 K_{11} K_{12} \sin \Omega_D \cos \Omega_D$$

$$+ K_{22}^{\ 2} \cos^2 \Omega_D) \cos \Omega_D$$

$$= K_{11} K_{12} \sin \Omega_D - K_{22}^{\ 2} \cos \Omega_D \qquad \text{(O-102)}$$

Since K_{12} is of order ϵ^2 and K_{22} is of order ϵ, the third factor is of order ϵ^2. Therefore, c_η, being proportional to the ratio of $(D \sin \Omega_D - B^2 \cos \Omega_D)$ to C, is of order ϵ, and hence for the special case

$$c_\eta = 0 \qquad \text{(O-103)}$$

The component c_ξ in Eq. (O-100) is

$$c_\xi = \frac{n \sin \Omega_D}{4 B} \delta q_D^-$$

$$= \frac{n (1 - e \cos E_D)}{6 N \pi (1 + e \cos E_D)} \delta q_D^-$$

$$= \frac{\mu}{3 N P a v_D^{\ 2}} \delta q_D^- \qquad \text{(O-104)}$$

The VTA velocity correction vector is

$$\underline{c}_v = \frac{\mu}{3 N P a v_D^{\ 2}} (\delta q_D^-) \underline{u}_{\xi_D} \qquad \text{(O-105)}$$

where $\underline{u}_{\xi_D}$ is a unit vector in the ξ_D direction.

Eq. (O-105) can be compared with Eq. (O-13), which is an expression for the FTA velocity correction for the same special case (i.e., $\delta \underline{r}_D^-$ is in the q_D direction). The magnitude of the correction is the same for both FTA and VTA guidance; hence there is no propellant saving when VTA guidance is used. A more interesting result is that the FTA correction is applied in the q_D direction, while the VTA correction is applied in the ξ_D direction. It is surprising, to say the least, that two corrections of the same magnitude but seemingly in different directions achieve the objectives of the two guidance schemes. This confusing state of affairs can be clarified by an investigation of the orientation of the vector $\underline{w}$.

$$\underline{w} = \overset{*}{K}_{CD} \underline{v}_R = \begin{Bmatrix} k_{11} v_{R_p} + k_{12} v_{R_q} \\ k_{21} v_{R_p} + k_{22} v_{R_q} \\ k_{33} v_{R_z} \end{Bmatrix}$$

$$= n \begin{Bmatrix} \dfrac{1}{X \sin E_M} \begin{pmatrix} K_{11} v_{R_p} + K_{12} v_{R_q} \\ K_{21} v_{R_p} + K_{22} v_{R_q} \end{pmatrix} \\ \hline \dfrac{1}{\sin (f_D - f_C)} \begin{pmatrix} K_{33} v_{R_z} \end{pmatrix} \end{Bmatrix} \tag{O-106}$$

When $(t_D - t_C)$ is very nearly equal to NP,

$$\underline{w} = \frac{n}{\epsilon} \left\{ \begin{array}{c} \dfrac{(-1)^N \, v_{R_p}}{2\,(1 - e \cos E_D)} \\[2ex] - \dfrac{(1 - e \cos E_D)\, \epsilon\, v_{R_q}}{6\, N \pi\, (1 + e \cos E_D)} \\[2ex] \dfrac{(1 - e^2)^{1/2}\, v_{R_z}}{(1 - e \cos E_D)^2} \end{array} \right\} \qquad \text{(O-107)}$$

The only term inside the braces in Eq. (O-107) that contains the infinitesimal ϵ is the term representing the component of $\underline{w}$ in the q_C direction. Thus, when $(t_D - t_C)$ is equal to N P, the $\underline{w}$ vector, although infinite in magnitude, must lie in the plane normal to the q_C-axis. But the q_C-axis and the q_D-axis coincide when $(t_D - t_C)$ equals N P. Therefore, regardless of the orientation of the relative velocity vector the noncritical vector $\underline{w}$ is perpendicular to the q_D-axis when $(t_D - t_C)$ equals N P. The ξ_D-axis has been defined as the axis normal to $\underline{w}$ and lying in the reference trajectory plane. Since both the q_D-axis and the ξ_D-axis are in the reference trajectory plane and perpendicular to $\underline{w}$, they must coincide, and consequently Eqs. (O-13) and (O-105) give the identical velocity correction for this special case.

Eq. (O-107) indicates that the p_C component of $\underline{w}$ changes sign on each successive circuit of the focus. This would appear to indicate that $\underline{w}$ rotates with a period that is twice the period P of the orbital motion. The rotation of $\underline{w}$ has not been investigated further in this study, but it is suggested as a possibly fruitful topic for future work in the field.

The third special case is the two-dimensional case, in which both δz_D^- and v_{R_z} are equal to zero. Under these conditions, the $\underline{v}_R$ vector lies in the reference trajectory plane. In the critical-plane coordinate system the ζ_D-axis lies along $\underline{v}_R$ in the reference trajectory plane, the ξ_D-axis is perpendicular to $\underline{v}_R$ and in the reference trajectory plane, and the η_D-axis coincides with the z-axis. Since δz_D^- is taken as zero, it follows that

$$\delta \eta_D^- = 0 \tag{O-108}$$

For a correction at $(t_D - t_C)$ equal to N P, it has been shown that the ξ_C-axis is the same as the q_D-axis. Thus, the miss distance vector $\delta \underline{\rho}^-$ must be parallel to the orbital velocity vector $\underline{v}_D$.

$$\delta \underline{\rho}^- = (\delta q_D^-) \ \underline{u}_{\xi_D} \tag{O-109}$$

Eq. (O-109) is the defining characteristic of the second special case. Therefore, both cases have the same solution for $\underline{c}_v$, which is given by Eq. (O-105).

It is of interest to note the difference in initial hypotheses between the second and the third special cases. In the second, the predicted position variation vector $\delta \underline{r}_D^-$ is assumed to be parallel to $\underline{v}_D$, and the orientation of the relative velocity vector $\underline{v}_R$ is arbitrary. In the third case, both $\delta \underline{r}_D^-$ and $\underline{v}_R$ lie in the reference trajectory plane, but each may have any arbitrary orientation in that plane.

The third case illustrates the pitfalls that may be encountered if the mathematical model is over-simplified. If a preliminary guidance study of a journey involving more than one circuit of the focus is based on a two-dimensional model, that is, a model in which both $\underline{v}_R$ and $\delta \underline{r}_D^-$ are assumed to lie in the reference trajectory plane, the analysis will indicate that a finite VTA correction can be computed when $(t_D - t_C) = N P$, whereas a three-dimensional model shows that, in general, such a computation is not possible.

O. 15 Effect on VTA Guidance of Singularities at $(f_D - f_C) = (2N-1)\pi$

When $(f_D - f_C) = (2N-1)\pi$, k_{33} is the only element of $\overset{*}{K}_{CD}$ that becomes infinite. The factors B and D in Eq. (O-86) are determined in routine fashion, and factor C reduces to

$$C = K_{33} X \sin E_M \cos i_D \tag{O-110}$$

The matrix $\overset{*}{Y}$ becomes

$$\overset{*}{Y} = \left\{ \begin{array}{ccc} & \left(\dfrac{1}{4B} \right. & \left. 0 \right) \\ \hline \dfrac{1}{X \sin E_M} & \left(\dfrac{D}{B} \right. & \left. -\dfrac{B}{\cos i_D} \right) \end{array} \right\} \tag{O-111}$$

A finite VTA velocity correction can be computed when $(f_D - f_C) = (2N - 1)\pi$ except for the special case when the relative velocity vector lies in the reference trajectory plane (that is, $\cos i_D = 0$).

The vector $\underline{w}$ corresponding to $(f_D - f_C) = (2N - 1)\pi$ may be expressed as

$$\underline{w} = \left\{ \begin{array}{c} k_{11} v_{R_p} + k_{12} v_{R_q} \\ \\ k_{21} v_{R_p} + k_{22} v_{R_q} \\ \\ \dfrac{n K_{33}}{\sin (f_D - f_C)} \; v_{R_z} \end{array} \right\} \tag{O-112}$$

Only the z-component of $\underline{w}$ goes to infinity in the singularity condition. Therefore, $\underline{w}$ is parallel to the z-axis, and, depending on the sign of v_{R_z},

$$i_C = 0° \text{ or } 180° \tag{O-113}$$

The ζ_C-axis is the z-axis, and the $\xi_C - \eta_C$ plane is the reference trajectory plane. Thus, the VTA correction vector must lie in the reference trajectory plane, regardless of the orientation of $\underline{v}_R$ (as long as $\cos i_D \neq 0$).

The equation for the VTA correction in the p q z coordinate system is obtained by substituting Eq. (O-112) into Eq. (M-10).

$$\underline{c}_V = - \begin{bmatrix} k_{11} & k_{12} & -\dfrac{k_{11}\, v_{R_p} + k_{12}\, v_{R_q}}{v_{R_z}} \\ k_{21} & k_{22} & -\dfrac{k_{21}\, v_{R_p} + k_{22}\, v_{R_q}}{v_{R_z}} \\ 0 & 0 & 0 \end{bmatrix} \delta \underline{r}_D^- \tag{O-114}$$

The z-component of the correction is zero, as required by the fact that the z-axis is the noncritical axis. The elements in the first two columns of the matrix in Eq. (O-114) are the same as the corresponding elements in the matrix of the equation for the FTA correction.

O. 16 Effect on VTA Guidance of Singularities at X = 0

For the singularities at X equal to zero,

$$C = B \sin (f_D - f_C) \sin i_D \tag{O-115}$$

The $\overset{*}{Y}$ matrix is

$$\overset{*}{Y} = n \left\{ \begin{array}{c|c} & \begin{pmatrix} \dfrac{1}{4B} & 0 \end{pmatrix} \\ \hline \dfrac{K_{33}}{\sin (f_D - f_C) \sin i_D} & \begin{pmatrix} \dfrac{D \cos i_D}{B^2} & -1 \end{pmatrix} \end{array} \right\} \tag{O-116}$$

Parameters B and D are computed from Eq. (O-83) and (O-88), respectively.

A finite VTA correction can be determined when this type of singularity occurs except for the special case when the relative velocity vector is parallel to the z-axis (that is, $\sin i_D = 0$).

When X = 0, the vector $\underline{w}$ can be written as

$$\underline{w} = \left\{ \begin{array}{c} \dfrac{n}{X \sin E_M} \begin{pmatrix} K_{11}\, v_{R_p} + K_{12}\, v_{R_q} \\ \\ K_{21}\, v_{R_p} + K_{22}\, v_{R_q} \end{pmatrix} \\ \hdashline \left(k_{33}\, v_{R_z} \right) \end{array} \right\} \tag{O-117}$$

Both w_p and w_q go to infinity, but w_z remains finite. Therefore, the $\underline{w}$ vector lies in the reference trajectory plane. The ξ_C - ζ_C plane is the reference trajectory plane, the η_C-axis is the z-axis, and

$$i_C = 90° \tag{O-118}$$

The direction of $\underline{w}$ in the reference trajectory plane varies with N, the number of circuits between t_C and t_D.

In the general case the correction component along the line of nodes at t_C (i.e., the ξ_C-component of $\underline{c}_W$) is affected by only that component of the miss distance lying along the line of nodes at t_D. Under conditions of the X = 0 singularity the ξ_C-component of the correction is the only correction component in the reference trajectory plane. Therefore, when X = 0, the entire correction component in the reference trajectory plane is due to only that component of the miss distance vector which lies along the line of nodes at t_D. The component of the miss distance that is in the reference trajectory plane but normal to the line of nodes must be compensated completely by the component of the correction that is parallel to the z-axis.

The $X = 0$ singularity does not affect the out-of-plane motion of the vehicle. The nature of the VTA correction will now be investigated for the case when position variation $\delta \underline{r}_D^-$ is parallel to the z-axis. The miss distance vector for this case is

$$(\delta \underline{\rho}^-)_W = \begin{Bmatrix} \delta \xi_D^- \\ \delta \eta_D^- \end{Bmatrix} = \begin{Bmatrix} 0 \\ \sin i_D \end{Bmatrix} \delta z_D^- \tag{O-119}$$

From Eqs. (O-75) and (O-116), the correction is

$$\underline{c}_W = \begin{Bmatrix} c_\xi \\ c_\eta \end{Bmatrix} = \begin{Bmatrix} 0 \\ -k_{33} \end{Bmatrix} \delta z_D^- \tag{O-120}$$

In vector form,

$$\underline{c}_v = - k_{33} (\delta z_D^-) \underline{u}_{\eta_C} = - k_{33} (\delta z_D^-) \underline{u}_z \tag{O-121}$$

Thus, when $X = 0$, a position variation in the z direction calls for a VTA correction in the z direction, and the motion in the reference trajectory plane is not affected.

O. 17 Physical Interpretation of the Effect of the Singularities on VTA Guidance

In the past three sections, it has been shown that, in general, it is not possible to compute a finite VTA velocity correction when a singularity of the first type occurs, i.e., when $(t_D - t_C) = NP$, and it is possible to compute a finite correction when either of the other two types of singularities occurs, i.e., when $(f_D - f_C) = (2N - 1)\pi$ or when $X = 0$. This capability is in contrast with the FTA method of guidance, in which no finite correction can generally be computed when any one of the three types of singularities occurs.

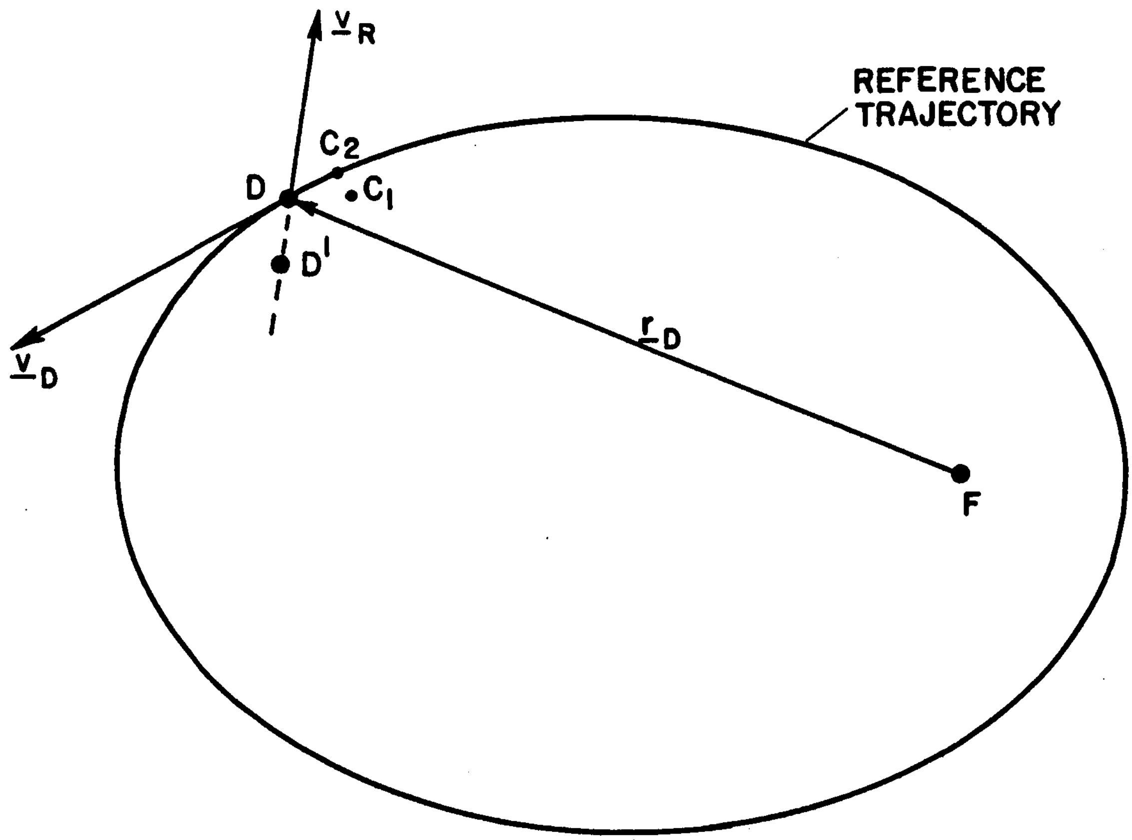
The key to a physical understanding of the difference between the two guidance concepts lies in the fact that FTA requires that the vehicle be at the specific point D at time t_D, while VTA has the less stringent requirement that the vehicle's position at $t = t_D$ be at any point near D on a specified straight line which passes through D.

In Fig. O.9, the relative velocity vector $\underline{v}_R$ is not in the reference trajectory plane. If at time $t_C = (t_D - NP)$ the vehicle is at some arbitrary point C_1 near D, the required VTA correction is such that the new trajectory contains the point C_1 and intersects the line of action of vector $\underline{v}_R$ at $t = t_D$. In the general three-dimensional case, it is not possible to find a trajectory which meets these requirements and at the same time differs only slightly from the reference trajectory. Therefore, the linear theory does not allow for the computation of a finite VTA velocity correction when $(t_D - t_C) = NP$.

In Section O.14, three special cases are considered. In the first, the vehicle's predicted position at $t = t_D$ is D', which lies along the line of action of $\underline{v}_R$. In this case the time at which a correction is contemplated is immaterial, since no correction is needed.

In the second special case, the vehicle's position at $t = t_C$ is C_2, which lies along the line of action of the orbital velocity vector $\underline{v}_D$. This case has already been taken up in Section O.5 in connection with FTA guidance. Irrespective of the nature of $\underline{v}_R$, the vehicle can be made to arrive at D at time t_D by applying a correction in the direction of $\underline{v}_D$ which causes the proper change in the period of the orbital motion.

The third special case occurs when the correction point C_1 and the vector $\underline{v}_R$ both lie in the reference trajectory plane. Then if $\underline{v}_R$ and $\underline{v}_D$ are not collinear, the trajectory passing through C_1 must cross the line of action of $\underline{v}_R$, and a small correction can be applied in the direction of $\underline{v}_D$ to ensure that such a crossing will take place at $t = t_D$. If $\underline{v}_R$ and $\underline{v}_D$ are collinear, the trajectory through C_1 does not cross the line of action of $\underline{v}_R$, and it is not possible to compute a small VTA velocity correction.



F – attractive focus

D – nominal destination point

D' – possible predicted position at nominal time of arrival at destination

C_1, C_2 – possible vehicle positions at time of correction

$\underline{v}_D$ – vehicle's nominal orbital velocity vector at time of arrival at destination

$\underline{v}_R$ – vehicle's nominal relative velocity with respect to destination planet at time of arrival

Figure O.9 VTA Guidance for Singularities at $t_D - t_C = NP$

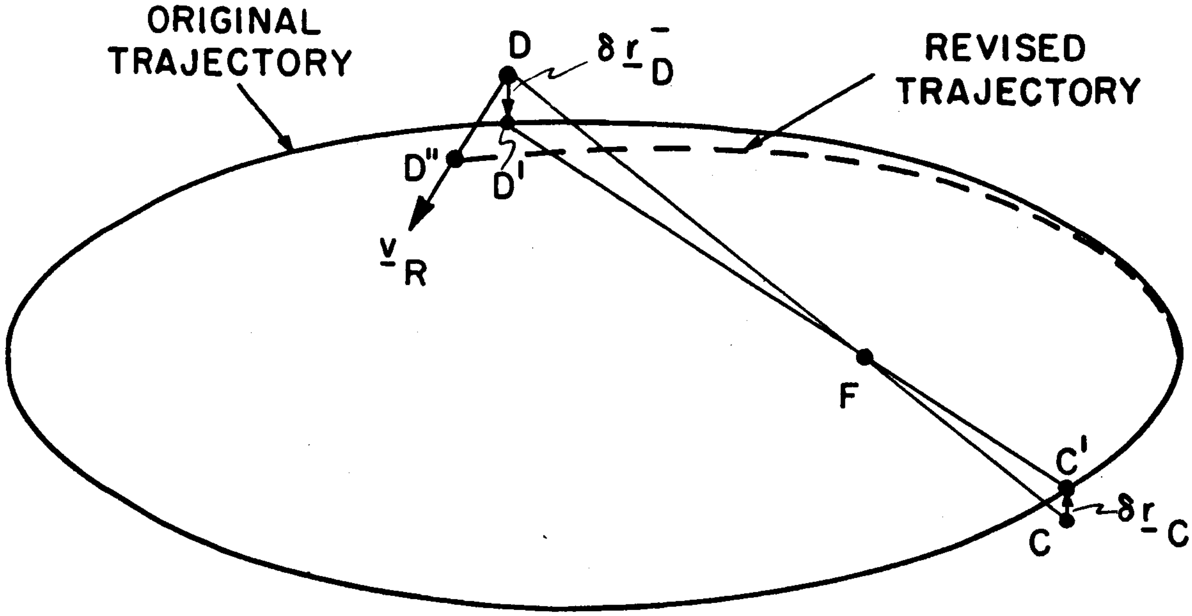
The second type of singularity, for which $(f_D - f_C) = (2N - 1)\pi$, is due to the vehicle's component of motion normal to the reference trajectory plane. If the z-component of $\delta \underline{r}_D^-$ can be effectively eliminated, a finite velocity correction can be computed. The VTA guidance concept provides a method of accomplishing this elimination as long as the relative velocity vector has a non-zero component in the z-direction.

In Fig. O.10, the vehicle's actual trajectory prior to any correction will cause it to be at point D' at $t = t_D$. The nominal destination point is D. The requirement of VTA guidance is that the vehicle's position at t_D lie along the line through D parallel to $\underline{v}_R$. Since $\underline{v}_R$ is assumed to have a non-zero z-component, a line through D parallel to $\underline{v}_R$ must intersect the plane of the actual trajectory; the point of intersection is D'' in the figure. The VTA guidance scheme computes the velocity correction required to get the vehicle to D'' at $t = t_D$. Thus, the correction is determined in such a way that the plane of the actual trajectory is not altered; the correction vector lies in the plane of the actual trajectory.

If $\delta z_D^- = 0$, the VTA correction computed by Eq. (O-114) is the same as the FTA correction which would be computed under the same circumstances. This is consistent with the argument that has just been presented since, when $\delta z_D^- = 0$, the points D and D'' coincide, and hence VTA and FTA corrections are identical.

The third type of singularity, for which $X = 0$, involves a fairly complex relationship between the eccentricity e and the eccentric anomalies E_C and E_D. When the delicate balance among these three quantities which must exist at $X = 0$ is upset by permitting some leeway in the choice of a time and place of arrival, it is reasonable to expect that the singularity will vanish and a finite VTA correction can be computed.

The $X = 0$ singularity is a characteristic of the motion in the plane of the reference trajectory. It is possible to use FTA guidance to compute a z-axis correction to a z-axis position variation even when $X = 0$. Comparison of Eq. (O-121) with Eq. (K-48) indicates that VTA and FTA systems yield the same z-axis correction to a z-axis position



F – attractive focus

C – position on reference trajectory at $t = t_C$

C' – position on actual original trajectory at $t = t_C$

D – position on reference trajectory at $t = t_D$

D' – position on actual original trajectory at $t = t_D$

D'' – position on revised trajectory at $t = t_D$

$\delta \underline{r}_C$ – position variation at $t = t_C$

$\delta \underline{r}_D$ – predicted position variation at $t = t_D$ before correction is applied

$\underline{v}_R$ – relative velocity vector of vehicle with respect to destination planet at $t = t_D$

Figure O.10 VTA Guidance for Singularities at $f_D - f_C = (2N-1)\pi$

variation when $X = 0$. Thus, if the predicted position variation at the destination is entirely in the z direction, the correction of smallest magnitude that can be made at a time for which $X = 0$ is the FTA correction corresponding to that time, and there is no change in the time of arrival. In this special case the component of position variation in the reference trajectory plane is zero, hence there is no need for the computation of the correction to become involved with the troublesome aspects of the $X = 0$ condition. The VTA system automatically takes this fact into consideration and provides a velocity correction which is parallel to the z-axis.

APPENDIX P

STATISTICAL THEORY

P.1 Summary

The components of a multi-dimensional random variable are to be estimated from a redundant set of measurements, associated with each of which there is some uncertainty. The estimation technique known as the method of maximum likelihood is used to make the estimate. The equations of the maximum likelihood method are developed in matrix form.

The concept of the equi-probability ellipsoid is introduced and is used as a quantitative indication of the accuracy of the estimate.

P.2 Introduction

The mathematical development of the method of maximum likelihood presented in the following sections is patterned after the work of Shapiro,[43] the primary difference being that in the case treated by Shapiro the likelihood equations are nonlinear, while in the present application they are linear. As a consequence, a closed-form solution is obtained in this appendix, whereas such a solution is not possible in the nonlinear case.

The method of maximum likelihood was originally developed by the British statistician R.A. Fisher. A rigorous mathematical treatment of the method is presented by Cramér.[44]

P.3 Mathematical Preliminaries

The number of measurements to be processed in the estimation procedure is designated as M. These measurements are collected in a single M-dimensional column vector $\underline{m}$. In the linear analysis the vector used in the computations is $\delta \underline{m}$, which consists of the variations of the components of $\underline{m}$ from their reference values. The reference values are computed *a priori.*

The parameters to be estimated are collected in the column vector $\underline{x}$. The linear analysis leads to an estimate of the variation of each of the components of $\underline{x}$ from its reference value. The variation in $\underline{x}$ is $\delta \underline{x}$. In the general case, $\delta \underline{x}$ is an N-dimensional vector, where N is any positive integer. For the problem of orbit determination N is equal to six.

For the i-th measurement the linear relationship between δm_i and $\delta \underline{x}$ can be expressed as the scalar product of the vector $\underline{q}_i$ and the vector $\delta \underline{x}$.

$$\delta m_i = \underline{q}_i^T \, \delta \underline{x} \tag{P-1}$$

$\underline{q}_i$ is a six-dimensional column vector whose components are the partial derivatives of m_i with respect to the components of $\underline{x}$. The partial derivatives are known functions of the parameter vector $\underline{x}$ and the time t_i. In the orbit determination problem, they can be expressed as functions of $\underline{r}_i$ and $\underline{v}_i$, the space vehicle's position and velocity vectors on the reference trajectory at time t_i.

The composite vector $\delta \underline{m}$ is obtained by extension of (P-1).

$$\delta \underline{m} = \begin{Bmatrix} \delta m_1 \\ \vdots \\ \delta m_M \end{Bmatrix} = \begin{Bmatrix} \underline{q}_1^T \\ \vdots \\ \underline{q}_M^T \end{Bmatrix} \delta \underline{x} = \overset{*}{Q}{}^T \, \delta \underline{x} \tag{P-2}$$

where the M-by-6 matrix $\overset{*}{Q}{}^T$ is defined by

$$\overset{*}{Q}{}^T = \begin{Bmatrix} \underline{q}_1^T \\ \vdots \\ \underline{q}_M^T \end{Bmatrix} \tag{P-3}$$

The transpose of $\overset{*}{Q}{}^T$ is the 6-by-M matrix $\overset{*}{Q}$.

$$\overset{*}{Q} = \left\{ \underline{q}_1 \cdots\cdots\cdots \underline{q}_M \right\} \tag{P-4}$$

The observed values of the measurements differ from the true values due to inaccuracies in instrumentation. If $\delta \underline{\tilde{m}}$ is the observed measurement variation vector, the measurement uncertainty vector $\underline{u}$ is defined by

$$\underline{u} = \delta \underline{\tilde{m}} - \delta \underline{m} \tag{P-5}$$

The covariance matrix of measurement uncertainties is

$$\overset{*}{U} = \overline{\underline{u}\ \underline{u}^T} \tag{P-6}$$

Each component of $\underline{u}$ is assumed to have a Gaussian probability distribution with zero mean. The elements of $\overset{*}{U}$ are determined a priori.

P.4 Conditional Probability Density

After a set of measurements has been made, the vector $\delta\underline{\tilde{m}}$ is known. The problem then is to estimate $\delta\underline{x}$ on the basis of the known $\delta\underline{\tilde{m}}$. The most probable value of $\delta\underline{x}$ is that value for which the conditional probability density $p(\delta\underline{x} \mid \delta\underline{\tilde{m}})$ is a maximum. $p(\delta\underline{x} \mid \delta\underline{\tilde{m}})$ is the probability density of the vector $\delta\underline{x}$ for the given measurement variation vector $\delta\underline{\tilde{m}}$.

Maximizing $p(\delta\underline{x} \mid \delta\underline{\tilde{m}})$ is not analytically feasible. However, the conditional probability density $p(\delta\underline{\tilde{m}} \mid \delta\underline{x})$ can be maximized; this probability density is known as the likelihood function $L(\delta\underline{x})$. The two conditional probability densities are related by the following equation:

$$L(\delta\underline{x}) = p(\delta\underline{\tilde{m}} \mid \delta\underline{x}) = \frac{p(\delta\underline{x} \mid \delta\underline{\tilde{m}}) \cdot p(\delta\underline{\tilde{m}})}{p(\delta\underline{x})} \tag{P-7}$$

$p(\delta\underline{\tilde{m}} \mid \delta\underline{x})$ is the conditional probability of obtaining the $\delta\underline{\tilde{m}}$ vector actually observed when the vector $\delta\underline{x}$ is specified. $p(\delta\underline{\tilde{m}})$ and $p(\delta\underline{x})$ are a priori probability densities.

The maximum likelihood estimate of $\delta\underline{x}$ is obtained by setting to zero the partial derivative of $L(\delta\underline{x})$ with respect to each component of $\underline{x}$ and then solving the resulting likelihood equations for the vector $\delta\underline{x}$. The maximum likelihood estimate of $\delta\underline{x}$ is designated $\delta\hat{\underline{x}}$.

From Equation (P-7) it is apparent that the maximum likelihood estimate and the most probable value of $\delta\underline{x}$ coincide if both $p(\delta\underline{\tilde{m}})$ and $p(\delta\underline{x})$ are independent of $\delta\underline{x}$.

P.5 The Maximum Likelihood Estimate

The observed measurement vector $\delta\underline{\tilde{m}}$ is the sum of a deterministic function of $\delta\underline{x}$ and the M-dimensional random variable $\underline{u}$.

$$\delta\underline{\tilde{m}} = \overset{*}{Q}^T\ \delta\underline{x} + \underline{u} \tag{P-8}$$

Therefore, the likelihood function becomes

$$L(\delta \underline{x}) = p(\delta \underline{\tilde{m}} \mid \delta \underline{x}) = p(\underline{u} \mid \delta \underline{x}) \tag{P-9}$$

The probability density of $\underline{u}$ is independent of $\delta \underline{x}$. Then,

$$L(\delta \underline{x}) = p(\underline{u}) \tag{P-10}$$

$p(\underline{u})$ represents the joint probability density of u_1, u_2, ..., u_M. The equation for the M-dimensional joint probability density is

$$p(\underline{u}) = p(u_1, \ldots \ldots, u_M)$$
$$= \frac{1}{\left[(2\pi)^M \left|\overset{*}{U}\right|\right]^{1/2}} \exp\left(- \frac{1}{2} \underline{u}^T \overset{*}{U}^{-1} \underline{u}\right) \tag{P-11}$$

where $\left|\overset{*}{U}\right|$ is the determinant of $\overset{*}{U}$.

Since $\log [p(\underline{u})]$ is a monotonically increasing function of $p(\underline{u})$, maximizing the logarithm yields the same value of $\delta \underline{x}$ as maximizing $p(\underline{u})$ itself. The mathematics is simplified slightly if $\log [p(\underline{u})]$ is the function that is maximized.

$$\log [p(\underline{u})] = - \frac{1}{2} \log\left[(2\pi)^M \left|\overset{*}{U}\right|\right]$$
$$- \frac{1}{2} \underline{u}^T \overset{*}{U}^{-1} \underline{u} \tag{P-12}$$

The first term on the right-hand side of (P-12) is a constant.

The partial derivative of $\log [p(\underline{u})]$ with respect to x_i, one of the components of $\underline{x}$, is

$$\frac{\partial \log [p(\underline{u})]}{\partial x_i} = - \frac{1}{2} \left(\frac{\partial \underline{u}^T}{\partial x_i} \overset{*}{U}^{-1} \underline{u} + \underline{u}^T \overset{*}{U}^{-1} \frac{\partial \underline{u}}{\partial x_i} \right) \tag{P-13}$$

The matrix product $\underline{u}^T \overset{*}{U}^{-1} \frac{\partial \underline{u}}{\partial x_i}$ is a scalar quantity, which is equal to its transpose. Since $\overset{*}{U}$ is a symmetric matrix, $\overset{*}{U}^{-1}$ is also symmetric. Then

$$\underline{u}^T \overset{*}{U}^{-1} \frac{\partial \underline{u}}{\partial x_i} = \left(\underline{u}^T \overset{*}{U}^{-1} \frac{\partial \underline{u}}{\partial x_i} \right)^T = \frac{\partial \underline{u}^T}{\partial x_i} \overset{*}{U}^{-1} \underline{u} \tag{P-14}$$

When (P-14) is substituted into (P-13) and (P-13) is equated to zero, the result is

$$\frac{\partial \underline{u}^T}{\partial x_i} \overset{*}{U}^{-1} \underline{u} = 0 \tag{P-15}$$

Since the observed measurement variation vector $\delta \underline{\tilde{m}}$ is independent of the components of $\delta \underline{x}$,

$$\frac{\partial \underline{u}^T}{\partial x_i} = \frac{\partial (\delta \underline{\tilde{m}} - \delta \underline{m})^T}{\partial x_i} = - \frac{\partial (\delta \underline{m})^T}{\partial x_i} \tag{P-16}$$

The expression on the right side of (P-16) is the negative of the elements composing the i-th row of the matrix $\overset{*}{Q}$, which is defined by Equation (P-4). The six equations corresponding to $i = 1, \ldots, 6$ in (P-15) can be combined into a single matrix equation

$$\overset{*}{Q} \overset{*}{U}^{-1} \underline{u} = \underline{0}_6 \tag{P-17}$$

In order to solve the set of simultaneous equations represented by (P-17) for the maximum likelihood estimate $\delta \underline{\hat{x}}$, it is necessary to relate $\underline{u}$ to $\delta \underline{\hat{x}}$. Although (P-5) defines $\underline{u}$ as being the difference between $\delta \underline{\tilde{m}}$ and $\delta \underline{m}$, a new vector $\delta \underline{\hat{m}}$ will now be defined, and $\underline{u}$ will be taken as the difference between $\delta \underline{\tilde{m}}$ and $\delta \underline{\hat{m}}$. $\delta \underline{\hat{m}}$ is the maximum likelihood estimate of the true measurement variation vector $\delta \underline{m}$.

$$\delta \underline{\hat{m}} = \overset{*}{Q}^T \delta \underline{\hat{x}} \tag{P-18}$$

$$\underline{u} = \delta \underline{\tilde{m}} - \delta \underline{\hat{m}} = \delta \underline{\tilde{m}} - \overset{*}{Q}^T \delta \underline{\hat{x}} \tag{P-19}$$

(P-17) and (P-19) are combined and solved for $\delta \underline{\hat{x}}$.

$$\delta \underline{\hat{x}} = (\overset{*}{Q} \overset{*}{U}^{-1} \overset{*}{Q}^T)^{-1} \overset{*}{Q} \overset{*}{U}^{-1} \delta \underline{\tilde{m}} \tag{P-20}$$

This is the matrix form of the equation for the maximum likelihood estimate of $\delta \underline{x}$ based on the observed measurements represented by $\delta \underline{\tilde{m}}$.

P.6 Uncertainty in the Maximum Likelihood Estimate

Let $\underline{\epsilon}$ be the difference between the estimate $\delta \underline{\hat{x}}$ and the true parameter variation vector $\delta \underline{x}$. $\underline{\epsilon}$ represents the uncertainty in the maximum likelihood estimate.

$$\underline{\epsilon} = \delta \underline{\hat{x}} - \delta \underline{x} \tag{P-21}$$

$\underline{\epsilon}$ can be written as a function of the measurement uncertainty vector $\underline{u}$ by performing a few simple matrix manipulations of (P-20).

$$\overset{*}{Q} \overset{*}{U}^{-1} \overset{*}{Q}^{T} \delta \underline{\hat{x}} = \overset{*}{Q} \overset{*}{U}^{-1} \delta \underline{\tilde{m}}$$

$$= \overset{*}{Q} \overset{*}{U}^{-1} (\delta \underline{m} + \underline{u}) \tag{P-22}$$

$$\overset{*}{Q} \overset{*}{U}^{-1} \overset{*}{Q}^{T} (\delta \underline{\hat{x}} - \delta \underline{x}) = \overset{*}{Q} \overset{*}{U}^{-1} \underline{u} \tag{P-23}$$

$$\underline{\epsilon} = (\overset{*}{Q} \overset{*}{U}^{-1} \overset{*}{Q}^{T})^{-1} \overset{*}{Q} \overset{*}{U}^{-1} \underline{u} \tag{P-24}$$

The covariance matrix $\overset{*}{E}$ of the vector $\underline{\epsilon}$ is

$$\overset{*}{E} = \overline{\underline{\epsilon} \ \underline{\epsilon}^{T}} = (\overset{*}{Q} \overset{*}{U}^{-1} \overset{*}{Q}^{T})^{-1} \overset{*}{Q} \overset{*}{U}^{-1} \overline{\underline{u} \ \underline{u}^{T}} \ \overset{*}{U}^{-1} \overset{*}{Q}^{T} (\overset{*}{Q} \overset{*}{U}^{-1} \overset{*}{Q}^{T})^{-1}$$

$$= (\overset{*}{Q} \overset{*}{U}^{-1} \overset{*}{Q}^{T})^{-1} \tag{P-25}$$

P.7 The Equi-Probability Ellipsoid

For an N-dimensional parameter estimate, the joint probability density of the components of the associated uncertainty vector $\underline{\epsilon}$ is

$$p(\underline{\epsilon}) = \frac{1}{\left[(2\pi)^{N} \left| \overset{*}{E} \right|\right]^{1/2}} \exp\left(- \frac{1}{2} \underline{\epsilon}^{T} \overset{*}{E}^{-1} \underline{\epsilon}\right) \tag{P-26}$$

Some useful results are obtained by setting the quadratic form in the argument of the exponential equal to a constant.

$$\underline{\epsilon}^T \overset{*}{E}^{-1} \underline{\epsilon} = k^2 \tag{P-27}$$

(P-27) is the equation of an N-dimensional ellipsoid centered at $\underline{\epsilon} = \underline{0}_N$. For a specified value of k, the joint probability density of any point on the ellipsoidal surface is

$$p_k(\underline{\epsilon}) = \frac{1}{\left[(2\pi)^N \, |\overset{*}{E}|\right]^{1/2}} \cdot \exp\left(-\frac{k^2}{2}\right) \tag{P-28}$$

Because the joint probability density is constant for all points on the surface, the ellipsoid of (P-27) is known as the equi-probability ellipsoid.

The equi-probability ellipsoid is a convenient means of comparing the accuracies obtained from various estimation methods. If for a given k the ellipsoid obtained by one estimation technique lies wholly inside the ellipsoid obtained by a second technique, the first technique obviously is more accurate than the second. If the ellipsoids derived from the two estimation methods intersect, the issue is not so clear-cut; depending on the distribution of the uncertainties in the measurements, either method may lead to a more accurate estimate of the parameter vector in a specific case.

If k^2 in Equation (P-27) is set equal to (N + 2), the resulting ellipsoid is known as the ellipsoid of concentration. This particular ellipsoid has the characteristic that, if the joint probability density is constant throughout the volume of the ellipsoid and zero everywhere outside the surface of the ellipsoid, the covariance matrix of the resulting distribution is the same as the covariance matrix $\overset{*}{E}$ of the original distribution.

Cramér has shown that there is a certain minimum size of the ellipsoid of concentration. Estimation techniques are compared on the basis of the ratios of the volumes of their ellipsoids of concentration to the volume of the minimum ellipsoid. For a linear process with Gaussian distribution of measurement uncertainties, the ellipsoid of concentration obtained by the method of maximum likelihood is equal to the minimum ellipsoid; therefore, the maximum likelihood estimate is an optimal estimate for such a case.

Another type of equi-probability ellipsoid that is frequently used in error analysis is that for which $k^2 = 1$. This type is known as the error ellipsoid. All equi-probability ellipsoids are geometrically similar. The ratio of the axis lengths of the ellipsoid of concentration to the corresponding axis lengths of the error ellipsoid is $(N + 2)^{1/2}$.

For a specified value of N, the probability that the uncertainty vector $\underline{\epsilon}$ falls completely within the error ellipsoid is a constant. For N = 2, the probability is 0.393; for N = 3, the probability is 0.199.

P.8 Circular Probable Error and Spherical Probable Error

Another type of equi-probability ellipsoid that is used in error analyses is the 50% probability ellipsoid, which is defined as the ellipsoid for which the probability is 0.5 that the vector $\underline{\epsilon}$ will lie totally within its boundaries. This concept is particularly useful when N is equal to 2 or 3, for in these cases it has a simple physical interpretation.

The volume of the N-dimensional equi-probability ellipsoid is

$$V = \frac{\pi^{\frac{N}{2}} k^N \left| \overset{*}{E} \right|^{1/2}}{\Gamma(\frac{N}{2} + 1)} \qquad \text{(P-29)}$$

where $\Gamma(\ )$ represents the gamma function of the argument.

For N = 2 the ellipsoid reduces to an ellipse, and its area is

$$A = \pi k^2 \left| \overset{*}{E} \right|^{1/2} \qquad \text{(P-30)}$$

The value of k for the 50% probability ellipse is 1.1774. The area of the 50% probability ellipse is

$$A_{0.5} = \pi (1.1774)^2 \left| \overset{*}{E} \right|^{1/2} \qquad \text{(P-31)}$$

When the measurements are carefully chosen, it is usually possible to obtain an ellipse whose two major axes are nearly equal in length. Then the ellipse closely resembles a circle. The radius of the circle with the same area as that given by (P-31) is known as the circular probable error (CPE). The CPE is frequently used as an accuracy criterion for two-dimensional parameter vectors. From (P-31),

$$\mathrm{CPE} = 1.1774 \; |\overset{*}{E}|^{1/4} \tag{P-32}$$

A similar criterion can be derived for N = 3. The volume of the three-dimensional equi-probability ellipsoid is

$$V = \frac{4}{3} \pi k^3 \; |\overset{*}{E}|^{1/2} \tag{P-33}$$

For the 50% probability ellipsoid, k = 1.5382.

$$V_{0.5} = \frac{4}{3} \pi \, (1.5382)^3 \, |\overset{*}{E}|^{1/2} \tag{P-34}$$

where $V_{0.5}$ is the volume of the 50% probability ellipsoid. When the axes of the ellipsoid are roughly equal in length, the spherical probable error (SPE) is defined as the radius of the sphere whose volume is equal to $V_{0.5}$.

$$\mathrm{SPE} = 1.5382 \, |\overset{*}{E}|^{1/6} \tag{P-35}$$

The numerical values used in the last two sections have been obtained from Burington and May[45] and from Locke.[46]